

# ON THE BIVARIATE ERDŐS-KAC THEOREM AND CORRELATIONS OF THE MÖBIUS FUNCTION

ALEXANDER P. MANGEREL

ABSTRACT. Let  $2 \leq y \leq x$  such that  $\beta := \frac{\log x}{\log y} \rightarrow \infty$ . Let  $\omega_y(n)$  denote the number of distinct prime factors  $p$  of  $n$  such that  $p \leq y$ , and let  $\mu_y(n) := \mu^2(n)(-1)^{\omega_y(n)}$ , where  $\mu$  is the Möbius function. We prove that if  $\beta$  is not too large (in terms of  $x$ ) then for each fixed  $a \in \mathbb{N}$ ,

$$\sum_{n \leq x} \mu_y(n) \mu_y(n+a) \ll x \left( \frac{1}{\log_2 y} + e^{-\frac{1}{21}\beta \log \beta} \right).$$

This can be seen as a partial result towards the binary Chowla conjecture. Our main input is a *quantitative* bivariate analogue of the Erdős-Kac theorem regarding the distribution of the pairs  $(\omega(n), \omega(n+a))$ , where  $n$  and  $n+a$  both belong to any subset of the positive integers with suitable sieving properties; moreover, we show that the set of squarefree integers is an example of such a set. We end with a further application of this probabilistic result related to a problem of Erdős and Mirsky on the number of integers  $n \leq x$  such that  $\tau(n) = \tau(n+1)$ .

## 1. INTRODUCTION

**1.1. On the Binary Chowla Conjecture.** Let  $\mu$  denote the Möbius function. It is well-known that the Prime Number Theorem is equivalent to the statement that  $\sum_{n \leq x} \mu(n) = o(x)$ . Thus,  $\mu$  exhibits a lot of cancellation, and we in fact expect (according to the density hypothesis) that for each  $\epsilon > 0$  and  $x$  sufficiently large,  $\mu$  has a sign change in any interval of the form  $(x, x+x^\epsilon]$ . We also expect that the sign changes are random, and do not exhibit conspiratorial tendencies, such as  $\mu(n)$  and  $\mu(n+a)$  frequently changing sign simultaneously for a given, fixed  $a \in \mathbb{N}$ . One of the first enunciations of this latter principle is due to Chowla [2].

**Conjecture 1.1** (Chowla). *Let  $k \geq 2$  and let  $0 \leq a_1 < a_2 < \dots < a_k$  be integers. Then*

$$(1) \quad \sum_{n \leq x} \mu(n+a_1) \cdots \mu(n+a_k) = o(x).$$

In the binary case, i.e.,  $k = 2$ , Chowla's conjecture is the statement that for any fixed  $a \in \mathbb{N}$ ,

$$(2) \quad \sum_{n \leq x} \mu(n) \mu(n+a) = o(x).$$

This is currently not known for *any*  $a \in \mathbb{N}$ . In fact, even to show that the left side of (2) has absolute value at most  $cx$ , where  $c < 1$ , was an intractable problem until very recently, when Matomäki and Radziwiłł were able to prove this as a consequence of their work on short averages of multiplicative functions (see Corollary 2 of [16] for the corresponding result with the Liouville function  $\lambda$  in place of  $\mu$ ).

There has been some remarkable recent progress on Conjecture 1.1 itself. We mention the two following notable examples. Tao [21] proved a logarithmically averaged version of (2), that is

$$(3) \quad \sum_{n \leq x} \frac{\mu(n)\mu(n+a)}{n} = o(\log x).$$

Unfortunately, (3) is a strictly weaker estimate than (2). In a different direction, Matomäki, Radziwiłł and Tao [17] proved that if one averages over all shift vectors  $\mathbf{a} \in [0, H]^k$ , where  $H \rightarrow \infty$  as  $x \rightarrow \infty$  then (2) holds for almost every such  $\mathbf{a}$ .

We shall prove the following two partial results in the direction of (2), which provide further motivation for the binary case of Conjecture 1.1 for any *fixed* shift.

Set  $B(x) := \frac{\sqrt{\log_2 x}}{\log_3 x \log_4^2 x}$ , where  $\log_k x$  is the  $k$ th iterated logarithm and  $x$  is sufficiently large.

**Theorem 1.2.** *a) Let  $2 \leq y \leq x$  and let  $\beta := \frac{\log x}{\log y}$ . Suppose that  $\beta \leq e^{B(x)}$ . For  $n \in \mathbb{N}$  let  $\omega_y(n)$  denote the number of primes  $p$  dividing  $n$  with  $p \leq y$ , and let  $\mu_y(n) := \mu^2(n)(-1)^{\omega_y(n)}$ . Then for each fixed  $a \in \mathbb{N}$ ,*

$$(4) \quad \sum_{n \leq x} \mu_y(n)\mu_y(n+a) \ll x \left( \frac{1}{\log_2 y} + e^{-\frac{1}{21}\beta \log \beta} \right).$$

*b) Let  $\mu(n; u) := \mu^2(n)e^{iu\omega(n)}$ . If  $|u|, |v| \leq 2\pi$  and  $w := \max\{|u|, |v|\}$ , we have*

$$(5) \quad \sum_{n \leq x} \mu(n; u)\mu(n+a; v) \ll x \left( w \log(1/w) + (\log x)^{-\frac{1}{3}(u^2+v^2)} \right).$$

Note that (4) states that if we only account for those prime factors  $p$  of integers  $n \leq x$  such that  $p \leq x^{o(1)}$  then the corresponding correlation sums of  $\mu_y$  are small. In other words, those sign changes of  $\mu(n)$  that are caused by "small primes" do not appear to correlate with those of  $\mu(n+a)$ . (5), while not directly related to (2), roughly shows that if we replace  $\mu(n) = \mu^2(n)e^{i\pi\omega(n)}$  by  $\mu^2(n)e^{iu\omega(n)}$  for *any*  $u = o(1)$  that is not too small as  $x \rightarrow \infty$  then the corresponding correlation sums are also small.

**Remark 1.3.** Results like (5) appear to be completely new, and provide a collection of examples in the direction of a general conjecture on the size of correlations of non-pretentious multiplicative functions, originally due to Elliott (see, for example, Conjecture 1.5 in [17]).

Results like (4) do exist in the literature, but in weaker forms. To the author's knowledge, the first result of this form is Theorem 5 in [1], where a correlation estimate of the type of (4) is established for a truncated version of the Liouville function, i.e.,  $\lambda_y(n) := (-1)^{\Omega_y(n)}$ , where  $\Omega_y(n)$  is the number of prime factors  $p$  of  $n$  with  $p \leq y$ , counted with multiplicity. However, the parameter  $y$  is restricted in such a way that  $y \leq (\log x)^{2+o(1)}$ .

Subsequently, Daboussi and Sarközy [3] established a more general theorem, applicable to the truncation of *any* 1-bounded multiplicative function, in which one may take any  $y \leq x^{o(1)}$ . As a particular case, they show that

$$\sum_{n \leq x} \mu_y(n)\mu_y(n+1) \ll x \left( \frac{1}{(\log y)^9} + e^{-\frac{\beta}{8}} \right).$$

Their result is not explicitly stated for shifts  $a$  other than 1, though this restriction appears to be merely technical. We emphasize, though, that our probabilistic view of the problem motivates the methods that we employ in this paper, and these are substantially different from those used in [3]. Among other things, in [3] Brun's sieve is used, while in the present paper we appeal to the Rosser-Iwaniec sieve instead (see Lemma 2.4 below). This difference accounts for the appearance of the additional factor of  $\log \beta$  in the second term in (4) above. In particular, when  $y \geq x^{\frac{1}{3 \log_3 x}}$  our estimate is superior to theirs. We believe that this improvement is worthwhile, considering that the main interest in a result like (4) is in its effectiveness when  $y$  is as close to  $x$  as possible in support of Conjecture 1.1.

We also note that a ternary extension of the result in [3] was worked out by Ganguli in his thesis [9]. This extension essentially follows the same method of proof as that in [3].

As our results suggest, the main focus of our arguments is on the behaviour of the pairs  $(\omega(n), \omega(n+a))$ , where  $n$  and  $n+a$  are squarefree. Indeed, observe that we can express (2) equivalently in the form

$$\sum_{n \leq x} \mu^2(n) \mu^2(n+a) e^{iu\omega(n)} e^{iv\omega(n+a)} = o(x),$$

where  $u = v = \pi$ . A moment's reflection (assuming one has probabilistic inclinations) suggests that the function of  $u$  and  $v$  resembles some variant of a characteristic function for a random vector. That is, suppose  $\mathbf{X} := (X_1, X_2)$  is a random vector on a fixed probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ , and let  $\sigma_{\mathbf{X}} := \mathbb{P} \circ \mathbf{X}^{-1}$  denote its law. We recall that the *characteristic function* of  $\mathbf{X}$  is the Fourier transform of  $\sigma_{\mathbf{X}}$ , i.e.,

$$\phi_{\mathbf{X}}(\mathbf{t}) := \mathbb{E}[e^{i\mathbf{X} \cdot \mathbf{t}}] = \int_{\mathbb{R}^2} e^{i\mathbf{u} \cdot \mathbf{t}} d\sigma_{\mathbf{X}}(\mathbf{u}).$$

Let  $S_a$  denote the set of integers  $n$  such that  $n$  and  $n+a$  are both squarefree. Choosing the finite probability space given by  $[1, x] \cap S_a$  with its power set and normalized counting measure, the characteristic function of the random vector  $n \mapsto (\omega(n), \omega(n+a))$  is

$$\tilde{\phi}_x(\mathbf{t}) := E^*(x; a)^{-1} \sum_{n \leq x} \mu^2(n) \mu^2(n+a) e^{i(\omega(n), \omega(n+a)) \cdot \mathbf{t}},$$

where  $E^*(x; a) := \sum_{n \leq x} \mu^2(n) \mu^2(n+a)$ . In particular, (2) is stating that  $\tilde{\phi}_x(\pi, \pi) = o(1)$ , as  $x \rightarrow \infty$ . In the next subsection, we shall discuss an analogue of this estimate that *is* achievable, by way of a quantitative, bivariate generalization of the Erdős-Kac theorem. This is the crucial input into the proof of Theorem 1.2.

**Remark 1.4.** In principle, our arguments extend to  $k$ -ary correlations of  $\mu_y$  and  $\mu(\cdot; u)$ , as well as to certain classes of unimodular multiplicative functions. We postpone such extensions to a separate paper.

**Remark 1.5.** In a sense, the approach we use as support for Chowla's conjecture is misguided. In light of an argument of Tao [22], it is known that the estimate

$$(6) \quad \sum_{n \leq x} \mu(n) \mu(an+2) \ll_{\epsilon} \frac{x}{(\log x)^{2+\epsilon}},$$

for each  $a \in \mathbb{N}$ , would be sufficient to prove the Twin Prime conjecture. The notorious Parity Problem in Sieve Theory prevents the Twin Prime conjecture from being tractable via Sieve methods alone; hence, it would seem that an attempt at proving Chowla's conjecture must also invoke additional parity barrier-breaking arguments. Our approach, as outlined in Section 2, involves generalizing a probabilistic framework of Kubilius in order to analyze joint distributions, and makes use of a composition of Rosser-Iwaniec sieves (see Lemma 2.4). Thus, our method is heuristically insufficient to provide a full proof of Chowla's conjecture. In spite of this, we are content to provide here a basic framework for correlation problems upon which further investigations may build.

**1.2. A Quantitative Bivariate Erdős-Kac Theorem.** Given  $n \leq x$ , let

$$\tilde{\omega}_x(n) := (\omega(n) - \log_2 x) / \sqrt{\log_2 x},$$

where  $\omega(n)$  is the number of distinct prime factors of  $n$ . Furthermore, for  $z \in \mathbb{R}$  let

$$H_x(z) := x^{-1} |\{n \leq x : \tilde{\omega}_x(n) \leq z\}|.$$

$H_x$  is the distribution function for  $\tilde{\omega}_x(n)$ , a centred and normalized version of  $\omega$ . It is well-known that if we define a collection  $\{\theta_p\}_p$  of indicator functions indexed by primes such that  $\theta_p(n) = 1$  if  $p|n$  and  $\theta_p(n) = 0$  otherwise then  $\omega(n) = \sum_{p \leq x} \theta_p(n)$  for every  $n \leq x$ , and the values of  $\theta_p$  and  $\theta_q$  are *asymptotically* independent as  $x \rightarrow \infty$ . In spite of this,  $\omega$  still behaves like a sum of *genuinely* independent random variables in the sense that it obeys a Central Limit Theorem. Indeed, this conclusion is furnished by the Erdős-Kac theorem, which states that as  $x \rightarrow \infty$ ,  $H_x \rightarrow \Phi$  almost everywhere, where

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}t^2} dt$$

is the distribution function of a Gaussian random variable. The rate of convergence of  $H_x$  to the normal distribution has also been studied in this connection, and a best-possible estimate for the  $L^\infty$  distance between  $H_x$  and  $\Phi$  was conjectured by LeVeque [15] and proven by Rényi and Turán [18] to be  $O(1/\sqrt{\log_2 x})$ . This result echoes the best-possible rate in the Lindeberg Central Limit theorem, given by the Berry-Esséen Theorem (see Chapter XVI of [7]).

One expects that since  $n$  and  $n+a$  share only finitely many common prime factors, the values of  $\tilde{\omega}_x(n)$  and  $\tilde{\omega}_x(n+a)$  are also asymptotically independent as  $x \rightarrow \infty$ . It is therefore reasonable to guess that a *bivariate* analogue of the Erdős-Kac theorem should hold, in which the distribution function  $H_x$  is replaced by the two-dimensional analogue

$$H'_x(z, z') := x^{-1} |\{n \leq x : \tilde{\omega}_x(n) \leq z, \tilde{\omega}_x(n+a) \leq z'\}|,$$

and the Gaussian limit distribution is replaced by the uncorrelated bivariate Gaussian distribution

$$\Phi_{(2)}(z, z') := \Phi(z)\Phi(z').$$

This was in fact proven by LeVeque [15], though his result does not yield an effective rate of decay for  $\|H'_x - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)}$ .

**Remark 1.6.** Actually, more general theorems than that of LeVeque regarding the existence and characterization of joint distributions of additive functions and their shifts appear in the literature; for a synthesis of these results, see Chapter VII of [14]. However, the methods employed to prove these results, e.g., the Crámer-Wald trick, are typically not useful for producing quantitative results. In a related though distinct vein, see Chapter V in [14] for quantitative results related to the distribution of sums  $\sum_{1 \leq j \leq k} f_j(n + a_j)$ , where each  $f_j$  is a real-valued, strongly multiplicative function.

In the sequel, we will concern ourselves with questions surrounding a restricted analogue of LeVeque's theorem. Specifically, let  $a \in \mathbb{N}$  be fixed, let  $E^*(x; a)$  and  $\tilde{\omega}_x$  be as above and for  $z, z' \in \mathbb{R}$  let

$$F_{x,a}(z, z') := E^*(x; a)^{-1} \left| \{n \leq x : \mu^2(n) = \mu^2(n + a) = 1, \tilde{\omega}_x(n) \leq z, \tilde{\omega}_x(n + a) \leq z'\} \right|.$$

We shall prove the following.

**Theorem 1.7.** *Let  $a \in \mathbb{N}$ . Then as  $x \rightarrow \infty$ ,*

$$\|F_{x,a} - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} \ll (\log_3 x)(\log_2 x)^{-\frac{1}{4}}.$$

We will actually prove a rather more general result, Theorem 2.3, which gives an  $L^\infty$  distance estimate as in Theorem 1.7 for a distribution function associated to pairs  $(\omega(n), \omega(n + a))$ , where  $n$  and  $n + a$  belong to *any* subset of  $\mathbb{N}$  with suitable sieve properties. Theorem 1.7 is a corollary of this (in this connection, see Proposition 2.2).

**Remark 1.8.** As in the univariate case, we expect that the best error term here is  $O(1/\sqrt{\log_2 x})$  (to see that this is best possible, see Lemma 5.2). As mentioned, Rényi and Turán [18] obtained this error term in the classical Erdős-Kac theorem using analytic methods from multiplicative number theory. Indeed, their proof depends crucially on the fact that the characteristic function of the distribution they consider is

$$u \mapsto x^{-1} \sum_{n \leq x} e^{iu\omega(n)},$$

the mean value of the *multiplicative* function  $n \mapsto e^{iu\omega(n)}$  for  $n \leq x$ . This can be estimated quite precisely for each  $u \in [0, 2\pi]$  by the Selberg-Delange method (see Chapter II.5 of [23]). Such techniques are not at our disposal, however, and our methods are not sufficiently powerful to yield the best-possible estimate.

**Remark 1.9.** As an application of the more general Theorem 2.3 below, one can deduce an asymptotic formula for the number of integers  $n \leq x$  (*without* the squarefree restriction) such that  $\omega(n) = k_1$  and  $\omega(n + a) = k_2$ , where each  $k_j$  is sufficiently close to  $\log_2 x$ , and  $a \in \mathbb{N}$  is fixed. We note in this connection that Goudout (see Théorème 3 in [10]) has recently proved an upper bound of the correct order of magnitude for the number of such integers that is uniform over all  $1 \leq k_1, k_2 \leq R \log_2 x$ , given any fixed  $R > 0$ .

It will be convenient for us to use a slight modification of the function  $\tilde{\phi}_x$  defined in the previous subsection. For  $u, v \in \mathbb{R}$  let

$$\phi_{x,a}(u, v) := E^*(x; a)^{-1} \sum_{n \leq x} \mu^2(n) \mu^2(n + a) e^{i(u\tilde{\omega}_x(n) + v\tilde{\omega}_x(n + a))},$$

In terms of  $\phi_{x,a}$ , (2) is equivalent to the statement that  $\phi_{x,a}(u, v) = o(1)$ , where  $u = v = \pi\sqrt{\log_2 x}$ . Note that for this choice of  $u$  and  $v$ ,  $\chi(u, v) = o(1)$ , so it would suffice to know that  $|\phi_{x,a}(u, v) - \chi(u, v)| = o(1)$  to prove (2). By a well-known theorem of Lévy, the convergence in distribution  $\|F_{x,a} - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} \rightarrow 0$  is equivalent to the *pointwise* convergence of the corresponding characteristic functions of these distributions, i.e.,  $|\phi_{x,a}(u, v) - \chi(u, v)| \rightarrow 0$  as  $x \rightarrow \infty$  for each  $u, v \in \mathbb{R}$ . The assertion that this continues to be true when  $u, v$  are chosen to *grow as a function of  $x$*  is a more subtle and difficult one. While we do not demonstrate that the convergence is uniform in general, we develop a method to prove an effective bound, in terms of  $u$  and  $v$ , for the distance between  $\phi_{x,a}(u, v)$  and  $\chi(u, v)$  in a range of  $u$  and  $v$  depending on  $x$  (that unfortunately does not include  $u = v = \pi\sqrt{\log_2 x}$ ). We shall use this bound in conjunction with a bivariate analogue of a smoothing lemma of Esséen (see Lemma 3.1) to prove Theorem 1.7.

**1.3. On a Problem of Erdős and Mirsky.** For a positive-valued multiplicative function  $f$  let

$$S_f(x) := |\{n \leq x : f(n) = f(n+1)\}|.$$

It is generally a difficult problem to even determine whether  $S_f(x)$  tends to infinity with  $x$ . Much of the literature on problems of this type relate to the case where  $f = \tau$ , the divisor function, and this shall be our focus as well. We thus henceforth write  $S(x)$  to mean  $S_\tau(x)$ .

Erdős and Mirsky [5] famously conjectured that  $\lim_{x \rightarrow \infty} S(x) = \infty$ , i.e., that there are infinitely many integers  $n$  such that  $\tau(n) = \tau(n+1)$ . Based on ideas of C. Spiro, Heath-Brown proved this conjecture in the affirmative, giving the lower bound  $S(x) \gg x(\log x)^{-7}$ . Erdős, Pomerance and Sarkőzy have made the following conjecture regarding the order of magnitude of  $S(x)$ .

**Conjecture 1.10** ([6]). *We have  $S(x) \asymp x(\log_2 x)^{-\frac{1}{2}}$ .*

The latter three authors proved that the upper bound in Conjecture 1.10 holds, and cite an heuristic argument due to Bateman and Spiro as further motivation for this conjecture. In the opposite direction, Hildebrand [13] has shown that  $S(x) \gg x(\log_2 x)^{-3}$ . See the beginning of Section 6 for our version of the Bateman-Spiro heuristic.

Implementing techniques related to those used to prove Theorem 1.2 and Theorem 2.3, we can prove a partial result in this direction. Let  $A(x) := 21 \frac{\log_3 x}{\log_4 x}$ .

**Theorem 1.11.** *Let  $2 \leq y \leq x$  such that if  $y = x^{\frac{1}{\beta}}$  then  $A(x) \leq \beta \leq e^{B(x)}$ . Let  $\tau_y(n)$  be the number of divisors  $d|n$  such that if  $p|d$  then  $p \leq y$ .*

$$|\{n \leq x : \tau_y(n) = \tau_y(n+1)\}| \gg \frac{x}{\sqrt{\log_2 x}}.$$

In Section 6 we actually prove a more general result on the number of  $n \leq x$  such that  $\tau(n) = 2^j \tau(n+1)$  in a range of  $j$  depending on  $x$  that includes  $j = 0$ . See Theorem 6.1 for a precise statement.

Our methods also extend to the more general question of estimating from below the number of integers  $n \leq x$  with  $\tau_y(n) = \tau_y(n+a)$  for  $a \in \mathbb{N}$ , and with more effort, to questions such as determining how often  $\tau_y(n+a_1) = \cdots = \tau_y(n+a_k)$ , for  $0 \leq a_1 < \cdots < a_k$  and  $k \geq 2$ .

**1.4. Notation and Conventions.** Throughout this paper,  $a, b, k, l, m, d$  will always stand for positive integers, and  $p, q$  will always denote primes. We write  $\log_k x$  to mean  $\log(\log_{k-1} x)$ , with  $\log_1 x = \log x$ , and  $x$  is always assumed to be sufficiently large so that these quantities are *positive*. We will frequently write  $\mathfrak{L}_y$  to mean  $\sqrt{\log_2 y}$  and  $\mathfrak{L} := \mathfrak{L}_x$ . We will also denote by  $\sigma$  the shift map  $\sigma(n) := n + 1$  defined on  $\mathbb{N}$ .

We shall employ the usual conventions of Probability Theory. We denote by  $\mathbb{P}$  a fixed probability measure on a measurable space  $(\Omega, \mathcal{B})$  on which a given random vector  $X : \Omega \rightarrow \mathbb{R}^n$  is defined. If  $S \subseteq \mathbb{R}^n$  is Borel measurable and  $X^{-1}(S) =: A \in \mathcal{B}$  then we will write  $\mathbb{P}(X \in S)$  in place of  $\mathbb{P}(A)$ . We write  $\mathcal{L}(X)$  to denote the measure  $\mathbb{P} \circ X^{-1}$ , which we call the *law* of  $X$ . Obviously, the law of any random vector is a probability measure on  $\mathbb{R}^n$  equipped with its Borel  $\sigma$ -algebra. We let  $\mathbb{E}X$  denote the expectation of  $X$ , i.e.,

$$\mathbb{E}X := \int_{\Omega} X d\mathbb{P} = \int_{\mathbb{R}^n} t d\mathcal{L}(X)(t).$$

Given two probability measures  $\nu_1$  and  $\nu_2$  on the measurable space  $(\Omega, \mathcal{B})$  we define the *total variation distance* between them by

$$d_{TV}(\nu_1, \nu_2) := \sup_{A \in \mathcal{B}} |\nu_1(A) - \nu_2(A)|.$$

## 2. A BIVARIATE KUBILIUS MODEL AND MULTIVARIATE POISSON APPROXIMATION

In this section we give a general framework to study the distribution of the vector  $(\omega(n), \omega(n+a))$ , where  $n$  and  $n+a$  are confined to subsets of  $\mathbb{N}$  with suitable properties. To be precise, we introduce the following definition.

**Definition 2.1.** Let  $a \in \mathbb{N}$ . Let  $S \subseteq \mathbb{N}$ . We say that  $S$  is *siftable* with respect to  $a$  if:  
a) we have

$$E(x; a) := \sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) \gg_a x,$$

where  $1_S$  is the indicator function for  $S$ , and: b) there exist a non-negative multiplicative function  $f = f_a$  and a real number  $\theta \in (0, 1)$  such that:

- i)  $0 \leq f(p) \leq \frac{1}{p}$  for each prime  $p$ ,
- ii) for each pair of coprime squarefree integers  $k_1, k_2$  with  $k_1 k_2 \leq x$  we have

$$(7) \quad \sum_{\substack{n \leq x \\ n \equiv 0(k_1), n+a \equiv 0(k_2)}} 1_S(n) 1_S(n+a) 1_{(n, a) = 1} = f(k_1 k_2) (E(x; a) + R(x; k_1, k_2)),$$

and the remainders  $R(x; k_1, k_2)$  satisfy

$$\sum_{k_1 k_2 \leq x^\theta} |R(x; k_1, k_2)| \ll \frac{x}{(\log x)^3}.$$

Finally, we will say that a siftable set is *regular* if  $\sum_{p \leq y} f(p) \gg \log_2 y$  for all  $y$  sufficiently large.

Without loss of generality, we can, and will, assume that  $f(p) = 0$  whenever  $p|a$ .

Trivially,  $S = \mathbb{N}$  is siftable and regular. Less trivially:

**Proposition 2.2.** *The set of squarefree integers is siftable and regular with respect to each fixed  $a \in \mathbb{N}$ .*

We shall prove this fact in Section 4, as it is crucial in our application to both Theorems 1.2 and 2.3. By a similar argument, the set of  $r$ -free integers for each  $r \geq 3$  is siftable. We leave the verification of this statement to the reader.

Given a siftable set  $S$  and a fixed  $a \in \mathbb{N}$ , let  $E(x; a)$  denote the number of  $n \leq x$  coprime to  $a$  such that both of  $n, n + a \in S$ , and let  $E^*(x; a)$  denote the number of such  $n \leq x$  without the coprimality condition. In this context, put  $\lambda(x) := \sum_{p \leq x} f(p)$  and

$$\tilde{\omega}_{x,S}(n) := \frac{\omega(n) - \lambda(x)}{\sqrt{\lambda(x)}},$$

and set

$$F_{x,S,a}(z, z') := E(x; a)^{-1} |\{n \leq x : n, n + a \in S, (n, a) = 1, \tilde{\omega}_{x,S}(n) \leq z, \tilde{\omega}_{x,S}(n + a) \leq z'\}|, \\ F_{x,S,a}^*(z, z') := E^*(x; a)^{-1} |\{n \leq x : n, n + a \in S, \tilde{\omega}_{x,S}(n) \leq z, \tilde{\omega}_{x,S}(n + a) \leq z'\}|.$$

For convenience, we will write  $F_x$  and  $F_x^*$  in place of  $F_{x,S,a}$  and  $F_{x,S,a}^*$ , respectively, whenever  $S$  and  $a$  are clearly defined. We shall establish an estimate for the rate of convergence of the limiting process  $F_{x,S,a}(z, z') \rightarrow \Phi(z)\Phi(z')$  as  $x \rightarrow \infty$ . In the statement below we denote by  $\sigma$  the shift map  $\sigma(n) := n + 1$ , for  $n \in \mathbb{N}$ , and we write  $\sigma^a$  to denote the composition of  $\sigma$  with itself  $a$  times.

**Theorem 2.3.** *Let  $a \in \mathbb{N}$  be fixed and let  $S$  be a siftable, regular set with respect to  $a$ . Then*

$$\|F_{x,S,a} - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} \ll (\log_3 x)(\log_2 x)^{-\frac{1}{4}}.$$

Moreover, if  $1_S$  is multiplicative and  $S$  is also siftable for each divisor of  $a$  then

$$\|F_{x,S,a}^* - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} \ll (\log_3 x)(\log_2 x)^{-\frac{1}{4}}.$$

In particular, the limiting distribution of the vectors  $(\tilde{\omega}_{x,S}, \tilde{\omega}_{x,S} \circ \sigma^a)$  in the set  $S \cap (S - a)$  is the uncorrelated bivariate Gaussian distribution.

Let  $S$  be a siftable set and let  $a \in \mathbb{N}$  be fixed. Let  $y := x^{\frac{1}{\beta}}$  where  $\beta \geq 1$ , and put  $P := P(y) := \prod_{\substack{p \leq y \\ p \nmid a}} p$ . Our goal is to approximate the vector  $(\omega, \omega \circ \sigma^a)$  on  $S \cap (S - a)$  by a discrete random vector. In this direction, it is sufficient to construct a measure against which there is a collection of *independent* random vectors  $(\tilde{\theta}_p, \tilde{\theta}_q)$  that are good approximations for the deterministic events  $\{n \leq x : n, n + a \in S, p|n, q|n + a\}$ , for  $p, q$  distinct primes dividing  $P$ .

Given coprime  $k_1, k_2|P$ , define the set

$$E_{k_1, k_2} := \{n \leq x : (n, a) = 1, n, n + a \in S, k_1|n, k_2|(n + a), (n, P/k_1) = (n + a, P/k_2) = 1\}.$$

We define a set function

$$\nu_y(E_{k_1, k_2}) = E(x; a)^{-1} |E_{k_1, k_2}|.$$

Note that the sets  $\{E_{k_1, k_2}\}_{\substack{k_1, k_2|P \\ (k_1, k_2)=1}}$  are mutually disjoint, and their union is precisely the set of all  $n \leq x$  coprime to  $a$  such that  $n, n + a \in S$ . We use a sieve to estimate each individual  $\nu_y(E_{k_1, k_2})$ .



**Lemma 2.4.** *Let  $x \geq 3$  and  $y \leq D_1, D_2 < x$ . Put  $s_j := \log D_j / \log y$ , for  $j = 1, 2$ . Then if  $k_1, k_2 | P$  are coprime,*

$$\begin{aligned} \nu_y(E_{k_1, k_2}) &= f(k_1)f(k_2) \prod_{p|P/k_1 k_2} (1 - 2f(p)) \left( 1 + O \left( \max_{j=1,2} \left\{ e^{-(1+o(1))s_j \log s_j} + \log^{-\frac{1}{6}} D_j \right\} \right) \right) \\ &\quad + O \left( \sum_{m_1 m_2 \leq k_1 k_2 D_1 D_2} f(k_1)f(k_2) |R(x; m_1, m_2)| \right). \end{aligned}$$

*Proof.* For  $j = 1, 2$ , let  $\Lambda_j^+ := \{\lambda_j^+(d)\}_{d|P, d \leq D_j}$  and  $\Lambda_j^- := \{\lambda_j^-(d)\}_{d|P, d \leq D_j}$  be upper and lower bound sieves, i.e., sequences of real numbers that satisfy

$$1 * \lambda_j^-(d) \leq 1 * \mu(d) \leq 1 * \lambda_j^+(d)$$

for each  $d|P$  with  $d \leq D_j$ . We take  $\lambda_j^+$  and  $\lambda_j^-$  to be the upper and lower bound sieve weights of Rosser-Iwaniec, respectively (see Chapter 11 of [8]).

We can compose these sieve weights to produce upper and lower bound sieves by defining the two-dimensional weights

$$\varphi^+(m, n) := (1 * \lambda_1^+)(m) (1 * \lambda_2^+)(n)$$

$$\varphi^-(m, n) := (1 * \lambda_1^-)(m) (1 * \lambda_2^+)(n) + (1 * \lambda_1^+)(m) (1 * \lambda_2^-)(n) - (1 * \lambda_1^+)(m) (1 * \lambda_2^+)(n).$$

That  $\varphi^+$  is an upper bound sieve weight is immediate; that  $\varphi^-$  is a lower bound sieve weight follows from the identity

$$\varphi^-(m, n) = (1 * \lambda_1^-)(m) (1 * \lambda_2^-)(n) - ((1 * \lambda_1^+)(m) - (1 * \lambda_1^-)(m)) ((1 * \lambda_2^+)(n) - (1 * \lambda_2^-)(n)).$$

Therefore, by Definition 2.1 with  $X = E(x; a)$ ,

(8)

$$|E_{k_1, k_2}| \leq X \sum_{\substack{d_1|P/k_1 \ d_2|P/k_2 \\ (k_1 d_1, k_2 d_2)=1}} \lambda_1^+(d_1) \lambda_2^+(d_2) f(k_1 k_2 d_1 d_2) + \sum_{\substack{d_1|P/k_1 \ d_2|P/k_2 \\ (k_1 d_1, k_2 d_2)=1}} f(k_1 k_2) |R(x; k_1 d_1, k_2 d_2)|$$

(9)

$$|E_{k_1, k_2}| \geq X \sum_{\substack{d_1|P/k_1 \ d_2|P/k_2 \\ (k_1 d_1, k_2 d_2)=1}} (\lambda_1^+(d_1) \lambda_2^-(d_2) + \lambda_1^-(d_1) \lambda_2^+(d_2) - \lambda_1^+(d_1) \lambda_2^+(d_2)) f(k_1 k_2 d_1 d_2)$$

(10)

$$- \sum_{\substack{d_1|P/k_1 \ d_2|P/k_2 \\ (k_1 d_1, k_2 d_2)=1}} f(k_1 k_2) |R(x; k_1 d_1, k_2 d_2)|.$$

Since  $f$  is multiplicative, for any of the sign pairs  $(\eta_1, \eta_2) \in \{(+, +), (+, -), (-, +)\}$ ,

$$\sum_{\substack{d_1|P/k_1 \ d_2|P/k_2 \\ (k_1 d_1, k_2 d_2)=1}} \lambda_1^{\eta_1}(d_1) \lambda_2^{\eta_2}(d_2) f(k_1 k_2 d_1 d_2) = f(k_1) f(k_2) \sum_{\substack{d_1|P/k_1 \\ (d_1, k_2)=1}} \lambda_1^{\eta_1}(d_1) f(d_1) \sum_{\substack{d_2|P/k_2 \\ (d_2, k_1 d_1)=1}} \lambda_2^{\eta_2}(d_2) f(d_2).$$

Put  $h_k(m) := f(m)1_{(m,k)=1}$ . Then by Theorem 11.12 of [8], we have

$$\begin{aligned}
& \sum_{\substack{d_1|P/k_1 \\ (d_1,k_2)=1}} \lambda_1^{\eta_1}(d_1)f(d_1) \sum_{\substack{d_2|P/k_2 \\ (d_2,k_1d_1)=1}} \lambda_2^{\eta_2}(d_2)f(d_2) = \sum_{d_1|P/k_1} \lambda_1^{\eta_1}(d_1)h_{k_2}(d_1) \sum_{d_2|P} \lambda_2^{\eta_2}(d_2)h_{k_1k_2d_1}(d_2) \\
&= \left(1 + O\left(e^{-s_2 \log s_2} + \log^{-\frac{1}{6}} D_2\right)\right) \sum_{d_1|P/k_1} \lambda_1^{\eta_1}(d_1)h_{k_2}(d_1) \prod_{p|P/(k_1k_2d_1)} (1 - f(p)) \\
&= \left(1 + O\left(e^{-(1+o(1))s_2 \log s_2} + \log^{-\frac{1}{6}} D_2\right)\right) \prod_{p|P/k_1k_2} (1 - f(p)) \sum_{d_1|P/k_1} \lambda_1^{\eta_1}(d_1)h_{k_2k_1}(d_1) \prod_{p|d_1} (1 - f(p))^{-1} \\
&= \left(1 + O\left(\max_{j=1,2} \left\{e^{-(1+o(1))s_j \log s_j} + \log^{-\frac{1}{6}} D_j\right\}\right)\right) \prod_{p|P/k_1k_2} (1 - f(p)) (1 - f(p)(1 - f(p))^{-1}) \\
&= \left(1 + O\left(\max_{j=1,2} \left\{e^{-(1+o(1))s_j \log s_j} + \log^{-\frac{1}{6}} D_j\right\}\right)\right) \prod_{p|P/k_1k_2} (1 - 2f(p)).
\end{aligned}$$

This last estimate together with (8) and (10) gives

$$\begin{aligned}
|E_{k_1,k_2}| &= \left(1 + O\left(\max_{j=1,2} \left\{e^{-(1+o(1))s_j \log s_j} + \log^{-\frac{1}{6}} D_j\right\}\right)\right) Xf(k_1)f(k_2) \prod_{p|P/k_1k_2} (1 - 2f(p)) \\
&\quad + O\left(\sum_{d_1 \leq D_1, d_2 \leq D_2} f(k_1k_2)|R(x; k_1d_1, k_2d_2)|\right).
\end{aligned}$$

Moreover, we can trivially bound the remainder by

$$\sum_{d_1 \leq D_1, d_2 \leq D_2} |R(x; k_1d_1, k_2d_2)| \leq \sum_{m_1m_2 \leq D_1D_2k_1k_2} |R(x; m_1, m_2)|.$$

This completes the proof.  $\square$

**Corollary 2.5.** *Fix  $k_1, k_2|P$  with  $k_1k_2 \leq x^{\frac{\theta}{2}}$ , where  $\theta$  is associated to  $S$ . Let  $y = x^{\frac{1}{\beta}}$  and  $D := \left(\frac{x}{k_1k_2}\right)^{\frac{\theta}{4}}$ . Then*

$$\nu_y(E_{k_1,k_2}) = \left(1 + O\left(e^{-\frac{\theta}{8}\beta \log \beta} + \log^{-\frac{1}{6}} x\right)\right) f(k_1)f(k_2) \prod_{p|P/k_1k_2} (1 - 2f(p)).$$

*Proof.* Choose  $D_1 = D_2 = D$  in Lemma 2.4 so that  $s_1 = s_2 \geq \theta\beta/8$ . Note that  $k_1k_2D_1D_2 = x^{\frac{\theta}{2}}(k_1k_2)^{1-\frac{\theta}{4}} \leq x^\theta$ . Thus, since  $S$  is siftable, the sum of the remainder terms in Lemma 2.4 can be estimated as

$$\sum_{m_1m_2 \leq k_1k_2D_1D_2} |R(x; m_1, m_2)| \leq \sum_{m_1m_2 \leq x^\theta} |R(x; m_1, m_2)| \ll \frac{x}{\log^3 x}.$$

Note moreover that

$$\prod_{p|P/k_1k_2} (1 - 2f(p)) \gg \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^2 \gg (\log x)^{-2},$$

so that  $x(\log x)^{-3} \ll (\log x)^{-1} \prod_{p|P/k_1k_2} (1 - 2f(p))$ . The lemma now follows from Lemma 2.4 upon dividing by  $E(x, a)$ , since  $E(x, a) \gg x$  by assumption.  $\square$

Let  $\mathfrak{S}$  denote the algebra of subsets of  $\mathbb{N} \cap [1, x]$  generated by the sets  $E_{k_1, k_2}$ , with  $k_1$  and  $k_2$  coprime divisors of  $P$ . As mentioned, the sets  $E_{k_1, k_2}$  are mutually disjoint. Thus, any  $A \in \mathfrak{S}$  can be written as a

$$A = \bigcup_{(k_1, k_2) \in C} E_{k_1, k_2},$$

where  $C$  is a set of pairs of coprime divisors of  $P$ . Define

$$\begin{aligned} \nu_y(A) &= \sum_{(k_1, k_2) \in C} \nu_y(E_{k_1, k_2}) \\ &= E(x; a)^{-1} \sum_{(k_1, k_2) \in C} \sum_{\substack{n \leq x \\ (n, P) = k_1, (n+a, P) = k_2}} 1_S(n) 1_S(n+a) 1_{(n, a)=1}. \end{aligned}$$

We construct a measure  $\sigma_y$  whose value, determined on the sets  $E_{k_1, k_2}$  generating  $\mathfrak{S}$ , is defined by

$$\sigma_y(E_{k_1, k_2}) = f(k_1) f(k_2) \prod_{p|P/k_1 k_2} (1 - 2f(p)),$$

where  $f$  is the multiplicative function in (7). By Corollary 2.5, whenever  $k_1 k_2 \leq x^{\frac{\theta}{2}}$  we have

$$(11) \quad \nu_y(E_{k_1, k_2}) = \left(1 + O\left(e^{-\frac{1}{6}\beta \log \beta} + \log^{-\frac{1}{6}} x\right)\right) \sigma_y(E_{k_1, k_2}).$$

The following lemma is inspired by the one-dimensional treatment of Kubilius' model given in Chapter 3 of [4].

Put  $R(x, y) := e^{-\frac{\theta}{8} \min\{\theta, \frac{1}{2}\} \beta \log \beta} + \log^{-\frac{1}{6}} x$ .

**Lemma 2.6.** *As defined,  $\sigma_y$  is a probability measure. Moreover, for any  $A \in \mathfrak{S}$  we have*

$$\nu_y(A) = (1 + O(R(x, y))) \sigma_y(A) + O(R(x, y)).$$

*Proof.* Recall that the collection  $\{E_{k_1, k_2}\}_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1}}$  partitions the set of  $n \leq x$  such that  $n, n+a \in S$  and  $(n, a) = 1$ . Thus, to establish that  $\sigma_y$  is a probability measure it suffices to show that

$$(12) \quad \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1}} \sigma_y(E_{k_1, k_2}) = \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1}} f(k_1) f(k_2) \prod_{p|P/k_1 k_2} (1 - 2f(p)) = 1.$$

This is a straightforward computation. Indeed, as  $P$  is squarefree,

$$\begin{aligned}
& \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1}} f(k_1) f(k_2) \prod_{p | P/k_1 k_2} (1 - 2f(p)) \\
&= \prod_{p | P} (1 - 2f(p)) \sum_{k_1 | P} f(k_1) \prod_{p | k_1} (1 - 2f(p))^{-1} \sum_{k_2 | P/k_1} f(k_2) \prod_{p | k_2} (1 - 2f(p))^{-1} \\
&= \prod_{p | P} (1 - 2f(p)) \sum_{k_1 | P} f(k_1) \prod_{p | k_1} (1 - 2f(p))^{-1} \prod_{p | P/k_1} (1 + f(p)(1 - 2f(p))^{-1}) \\
&= \prod_{p | P} (1 - 2f(p)) (1 + f(p)(1 - 2f(p))^{-1}) \sum_{k_1 | P} f(k_1) \prod_{p | k_1} (1 - 2f(p))^{-1} (1 + f(p)(1 - 2f(p))^{-1})^{-1} \\
&= \prod_{p | P} (1 - f(p)) (1 + f(p)(1 - f(p))^{-1}) = 1.
\end{aligned}$$

We next establish that

$$(13) \quad \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1, k_1 k_2 > x^{\frac{\theta}{2}}}} \sigma_y(E_{k_1, k_2}) \ll R(x, y).$$

Assume that (13) has been proven. Note then that since  $\nu_y$  is a probability measure, (11) and (12) together imply that

$$\begin{aligned}
\sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1, k_1 k_2 > x^{\frac{\theta}{2}}}} \nu_y(E_{k_1, k_2}) &= 1 - \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1, k_1 k_2 \leq x^{\frac{\theta}{2}}}} \nu_y(E_{k_1, k_2}) \\
&= 1 - (1 + O(R(x, y))) \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1, k_1 k_2 \leq x^{\frac{\theta}{2}}}} \sigma_y(E_{k_1, k_2}) \\
&= \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1, k_1 k_2 > x^{\frac{\theta}{2}}}} \sigma_y(E_{k_1, k_2}) + O(R(x, y)) = O(R(x, y)).
\end{aligned}$$

Thus, if  $A = \bigcup_{(k_1, k_2) \in C} E_{k_1, k_2}$  and we write

$$C^+ := C \cap \{k_1, k_2 | P : (k_1, k_2) = 1, k_1 k_2 > x^{\frac{\theta}{2}}\}$$

then by (13),

$$\begin{aligned}
\nu_y(A) &= \sum_{(k_1, k_2) \in C} \nu_y(E_{k_1, k_2}) \\
&= (1 + O(R(x, y))) \sum_{(k_1, k_2) \in C} \sigma_y(E_{k_1, k_2}) + \sum_{(k_1, k_2) \in C^+} (\nu(E_{k_1, k_2}) - (1 + O(R(x, y))) \sigma_y(E_{k_1, k_2})) \\
&= (1 + O(R(x, y))) \sigma_y(A) + O \left( \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2) = 1, k_1 k_2 > x^{\frac{\theta}{2}}}} (\sigma_y(E_{k_1, k_2}) + \nu_y(E_{k_1, k_2})) \right) \\
&= (1 + O(R(x, y))) \sigma_y(A) + O(R(x, y)),
\end{aligned}$$

as claimed. Thus, it remains to prove (13).

To this end we use Rankin's trick. Let  $\epsilon > 0$  be sufficiently small but fixed. First, by assumption  $f(p) \leq \frac{1}{p}$  when  $p \geq 2$ . Thus,

$$f(p)^{1-\epsilon} - f(p) \leq \frac{1}{p} (p^\epsilon - 1) \ll \epsilon y^\epsilon \frac{\log p}{p},$$

for each  $p \leq y$  once  $\epsilon < 1/2$ . Hence,

$$\begin{aligned} \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1, k_1 k_2 > x^{\frac{\theta}{2}}}} \sigma_y(E_{k_1, k_2}) &= \prod_{\substack{p \leq y \\ p \nmid a}} (1 - 2f(p)) \sum_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1, k_1 k_2 > x^{\frac{\theta}{2}}}} f(k_1) f(k_2) \prod_{q | k_1 k_2} (1 - 2f(q)) \\ &\leq \prod_{\substack{p \leq y \\ p \nmid a}} (1 - 2f(p)) x^{-\frac{\epsilon\theta}{2}} \left( \sum_{k | P} f(k)^{1-\epsilon} \prod_{q | k} (1 - 2f(q))^{-1} \right)^2 \\ &\ll_\epsilon \prod_{\substack{p \leq y \\ p \nmid a}} (1 - 2f(p)) x^{-\frac{\epsilon\theta}{2}} \exp \left( 2 \sum_{\substack{q \leq y \\ q \nmid a}} f(q)^{1-\epsilon} (1 - 2f(p))^{-1} \right) \\ &\ll x^{-\frac{\epsilon\theta}{2}} \exp \left( 2 \sum_{p \leq y} (f(p)^{1-\epsilon} - f(p)) \right) \\ &\leq \exp \left( \epsilon \left( 2y^\epsilon \sum_{p \leq y} \frac{\log p}{p} - \frac{\theta}{2} \log x \right) \right). \end{aligned}$$

By Mertens' theorem, the exponential is  $\ll (y^\epsilon)^{2y^\epsilon} x^{-\frac{\epsilon\theta}{2}}$ . Choosing  $\epsilon := \frac{C}{\log y}$  for  $C > 0$  to be chosen, it follows that

$$\sum_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1, k_1 k_2 > x^{\frac{\theta}{2}}}} \sigma_y(E_{k_1, k_2}) \ll \exp \left( 2C \left( e^C - \frac{\theta \log x}{4 \log y} \right) \right) = \exp \left( 2C \left( e^C - \frac{\theta}{4} \beta \right) \right).$$

Put  $\lambda := \frac{\theta}{4} \beta$ . Choosing  $C := \log \lambda - \log_2 \lambda$  yields

$$\sum_{\substack{k_1, k_2 | P \\ (k_1, k_2)=1, k_1 k_2 > x^{\frac{\theta}{2}}}} \sigma_y(E_{k_1, k_2}) \ll e^{-\frac{2}{3} \lambda \log \lambda} \ll e^{-\frac{\theta}{6} \beta \log \beta}$$

and (13) follows.  $\square$

For each  $p \leq y$  let  $X_p := (\tilde{\theta}_p, \tilde{\theta}'_p) : [1, x]^2 \rightarrow \{0, 1\}$  denote the independent Bernoulli random vector satisfying

$$X_p = \begin{cases} (1, 0) & \text{Prob} = f(p) \\ (0, 1) & \text{Prob} = f(p) \\ (0, 0) & \text{Prob} = 1 - 2f(p). \end{cases}$$

Put  $\Sigma(y) := \sum_{p \leq y} X_p$ , and note that  $\sigma_y$  is the law of  $\Sigma(y)$ . Indeed, we can identify the event  $\{\Sigma(y) = (l_1, l_2)\}$  with the set

$$\mathcal{S}(l_1, l_2) := \bigcup_{\substack{k_1, k_2 | P \\ \omega(k_j) = l_j}}^* E_{k_1, k_2},$$

where the asterisk indicates a union over  $(k_1, k_2) = 1$ . Since  $X_p$  and  $X_q$  are independent in the probability space  $([1, x], \mathfrak{S}, \sigma_y)$ ,

$$\begin{aligned} \sigma_y(E_{k_1, k_2}) &= \prod_{p|k_1 k_2} f(p) \prod_{q|P/k_1 k_2} (1 - 2f(q)) \\ &= \prod_{p|k_1} \mathbb{P}(X_p = (1, 0)) \prod_{q|k_2} \mathbb{P}(X_q = (0, 1)) \prod_{r|P/k_1 k_2} \mathbb{P}(X_r = (0, 0)) \\ &= \mathbb{P}(X_p = (1, 0) \text{ iff } p|k_1, X_q = (0, 1) \text{ iff } q|k_2) = \mathbb{P}(\tilde{\theta}_p = 1 \text{ iff } p|k_1, \tilde{\theta}'_p = 1 \text{ iff } p|k_2). \end{aligned}$$

Therefore, it follows that

$$\sigma_y(\mathcal{S}(l_1, l_2)) = \sum_{\substack{k_1, k_2 | P \\ \omega(k_j) = l_j}}^* \sigma_y(E_{k_1, k_2}) = \sum_{\substack{k_1, k_2 | P \\ \omega(k_j) = l_j}}^* \mathbb{P}(\tilde{\theta}_p = 1 \text{ iff } p|k_1, \tilde{\theta}'_p = 1 \text{ iff } p|k_2) = \mathbb{P}(\Sigma(y) = (l_1, l_2)).$$

We note that since  $\sigma_y$  is an atomic measure, the relationship between  $\sigma_y$  and  $\nu_y$  in Lemma 2.6 continues to hold when both  $\sum_{p \leq y} X_p$ , and  $(\omega_y, \omega_y \circ \sigma^a)$  are respectively centred and normalized by  $\sum_{p \leq y} f(p)$ .

To proceed, we will require the following result due to Roos [19]. In what follows, fix  $d \in \mathbb{N}$  and for each  $1 \leq j \leq d$  let  $\mathbf{e}_j \in \mathbb{R}^d$  be the unit vector whose only non-zero component is in the  $j$ th position.

**Lemma 2.7** ([19], Corollary 1). *Let  $\{X_k\}_{k \geq 1}$  be a sequence of Bernoulli random  $d$ -vectors. Let  $\mathbb{P}(X_k = \mathbf{e}_j) = p_{k,j}$ ,  $p_k := \sum_{1 \leq j \leq d} p_{k,j}$ , and suppose that  $\mathbb{P}(X_k = \mathbf{0}) = 1 - p_k$ . Put  $S_n := \sum_{1 \leq k \leq n} X_k$  and  $\lambda_n(j) := \sum_{1 \leq k \leq n} p_{k,j}$ . Then if  $Z \sim \text{Poi}(\lambda_n(1), \dots, \lambda_n(d))$  with pairwise independent coordinates,*

$$d_{TV}(\mathcal{L}(S_n), \mathcal{L}(Z)) \ll \sum_{j \leq d} \min\{1, \lambda_n(j)^{-1}\} \sum_{1 \leq k \leq n} p_{k,j}^2.$$

An immediate consequence of our foregoing analysis and Lemma 2.7 is the following.

**Lemma 2.8.** *Let  $Z = (Z_1, Z_2) \sim \text{Poi}(\lambda(y), \lambda(y))$ , where  $\lambda(y) := \sum_{p \leq y} f(p)$ . Then*

$$d_{TV}(\nu_y, \mathcal{L}(Z)) \ll \frac{1}{\lambda(y)} + R(x, y).$$

*Proof.* By the triangle inequality, we have  $d_{TV}(\nu_y, \mathcal{L}(Z)) \leq d_{TV}(\nu_y, \sigma_y) + d_{TV}(\sigma_y, \mathcal{L}(Z))$ . Lemma 2.6 implies that  $d_{TV}(\nu_y, \sigma_y) \ll R(x, y)$ . We apply Lemma 2.7 with  $n = \pi(y)$  and  $p_{q,1} = p_{q,2} = f(q)$  for each  $q \leq y$ . Thus,  $\lambda_n(1) = \lambda_n(2) = \sum_{q \leq y} f(q)$ . Moreover, as  $f(q) \leq 1/q$  for each prime  $q$ ,  $\sum_{q \leq y} p_{q,j}^2 \ll 1$  for  $j = 1, 2$ . Thus, Lemma 2.7 indeed implies that  $d_{TV}(\sigma_y, \mathcal{L}(Z)) \ll \lambda(y)^{-1}$ , and the proof is complete.  $\square$

### 3. UNIFORM APPROXIMATION OF $\phi_x$ BY NORMAL AND POISSON CHARACTERISTIC FUNCTIONS

In this section, we show that  $\phi_{x,y}$  uniformly approximates the characteristic function of  $\Phi_{(2)}$ . We use this data to establish and estimate for the  $L^\infty$  distance between  $F_{x,S,a}$  and  $\Phi_{(2)}$ . The tool that makes this connection possible is the following two-dimensional version of a smoothing lemma due to Esséen (see Chapter XVI, Section 6 in [7]).

**Lemma 3.1.** *Let  $T \geq 1$ . Let  $G : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a differentiable function with  $\|\nabla G\|_\infty \leq 1$ , and such that  $G(t, t') \rightarrow 1$  as  $t, t' \rightarrow \infty$  and  $G(t, t') \rightarrow 0$  as  $t, t' \rightarrow -\infty$ . Let  $\chi$  be the Fourier transform of the Lebesgue-Stieltjes measure  $dG$ . Let  $H$  be the distribution function of a bivariate random vector whose characteristic function is  $\phi$ . Furthermore, suppose that for any fixed  $\mathbf{z}$ ,*

$$(14) \quad \left| G(\mathbf{z}) - H(\mathbf{z}) - \int_{-T}^{z_1} \int_{-T}^{z_2} d(G - H)(\mathbf{w}) \right| \ll T^{-1}.$$

Then

$$(15) \quad \|G - H\|_{L^\infty(\mathbb{R}^2)} \ll \int_{\mathcal{R}_T} \frac{|\phi(t, t') - \chi(t, t')|}{|tt'|} dt dt' + T^{-1},$$

where  $\mathcal{R}_T := \{\mathbf{u} \in [-T, T]^2 : T^{-3} < |u_1|, |u_2| \leq T\}$ .

The proof proceeds along similar lines to Esséen's one-dimensional lemma, but the author did not find it proved in the context of bivariate distribution functions. Thus, we prove it here for completeness.

*Proof.* Put  $\Delta := H - G$  and let  $\mathbf{z}_0 \in \mathbb{R}^2$  be such that  $\Delta(\mathbf{z}_0) = \|\Delta\|_{L^\infty(\mathbb{R}^2)}$  (we may assume positivity by defining  $\Delta$  as  $G - H$  instead, if necessary). Put  $w_T(y) := \frac{1 - \cos(Ty)}{\pi y^2}$  and let  $W_T(u, v) := w_T(u)w_T(v)$ . Note that

$$\hat{W}(r, s) = \hat{w}(r)\hat{w}(s) = \begin{cases} T^2 \left(1 - \frac{|r|}{T}\right) \left(1 - \frac{|s|}{T}\right) & \text{if } \max\{|r|, |s|\} \leq T \\ 0 & \text{otherwise.} \end{cases}$$

Also, write  $\Delta_T := \Delta * W_T$ . Let  $h_1, h_2 \geq 0$  and  $\mathbf{h} := (h_1, h_2)$ . Note that since  $H$  is a distribution function,  $H(\mathbf{z}_0) \geq H(\mathbf{z}_0 - \mathbf{h})$ . Thus, by Taylor's theorem,

$$\Delta(\mathbf{z}_0) - \Delta(\mathbf{z}_0 - \mathbf{h}) \geq H(\mathbf{z}_0) - H(\mathbf{z}_0 - \mathbf{h}) - |G(\mathbf{z}_0 - \mathbf{h}) - G(\mathbf{z}_0)| \geq -|(\nabla G)(\mathbf{h}) \cdot \mathbf{h}| \geq -(h_1 + h_2)$$

Now, let  $\delta := \Delta(\mathbf{z}_0)$ , let  $B := B_\delta(\mathbf{z}_0)$  be the  $\delta$ -ball (with respect to the  $L^\infty$ -norm), and put  $\mathbf{z} := \mathbf{z}_0 - \delta \cdot (1, 1)$ . Then for all  $\mathbf{u} \in B$ , we have  $H(\mathbf{z} + \mathbf{u}) \leq H(\mathbf{z}_0)$ . Noting that  $w_T$  is even, we have

$$\begin{aligned} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \Delta(\mathbf{z} + \mathbf{h}) W_T(\mathbf{h}) d\mathbf{h} &\leq \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} (\Delta(\mathbf{z}_0) - 2\delta + h_1 + h_2) W_T(\mathbf{h}) d\mathbf{h} \\ &= -4\delta \left( \int_0^{\delta} w_T(u) du \right)^2 = -4\delta T^2 \left( \int_0^{T\delta} \frac{1 - \cos u}{\pi u^2} du \right); \end{aligned}$$

this last statement follows by definition of  $\delta$  and because  $h_j w_T(h_j)$  is odd. Moreover,

$$\begin{aligned} \int_{\|\mathbf{h}\|_\infty > \delta} \Delta(\mathbf{z} + \mathbf{h}) W_T(\mathbf{h}) &\leq 4\delta \left( \int_0^\infty \frac{1 - \cos(Tx)}{\pi x^2} dx \right) \int_{|t| > \delta} \frac{1 - \cos(Tx)}{\pi x^2} \\ &= 4\delta T^2 \int_{u > T\delta} \frac{1 - \cos u}{\pi u^2} du \leq \frac{4}{\pi^2} T. \end{aligned}$$

Thus, we have

$$\begin{aligned} |\Delta_T(\mathbf{z})| &\geq \left| \int_{-\delta}^\delta \int_{-\delta}^\delta \Delta(\mathbf{z} + \mathbf{h}) W_T(\mathbf{h}) d\mathbf{h} \right| - \frac{4}{\pi^2} T \\ (16) \quad &\geq 4T^2 \left( \delta \left( \int_0^{T\delta} \frac{1 - \cos u}{\pi u^2} du \right)^2 - \frac{1}{\pi^2 T} \right) \\ (17) \quad &\geq 4T^2 \left( \delta - \frac{3}{\pi^2 T} \right) = 4T^2 \left( \|\Delta\|_\infty - \frac{3}{\pi^2 T} \right). \end{aligned}$$

It remains to show that  $|\Delta_T(\mathbf{z})|$  is bounded by the integral on the right side of (15). Now, let  $d\mu_T := W_T d\lambda$ , where  $d\lambda$  is Lebesgue measure on  $\mathbb{R}^2$ . Let  $B(\mathbf{z})$  denote the infinite box  $(-\infty, z_1] \times (-\infty, z_2]$ . Then the convolution of measures  $\Delta * \mu_T$  can be written as

$$\Delta * \mu_T(B(\mathbf{z})) = \int_{\mathbb{R}^2} \Delta(\mathbf{z} + \mathbf{u}) d\mu_T(\mathbf{v}) = \int_{\mathbb{R}^2} \Delta(\mathbf{z} + \mathbf{u}) W_T(\mathbf{v}) d\mathbf{v} = \Delta_T.$$

Now, by Fourier inversion, we have

$$\begin{aligned} \Delta * \mu_T(B(\mathbf{z})) &= \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} d(\Delta * \mu_T)(\mathbf{w}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} d\xi_1 d\xi_2 \left( \int_{\mathbb{R}^2} e^{-i\mathbf{u} \cdot \boldsymbol{\xi}} \hat{w}_T(u_1) \hat{w}_T(u_2) (\phi(\mathbf{u}) - \chi(\mathbf{u})) du_1 du_2 \right) \\ &= \frac{T^2}{2\pi} \lim_{\tau_1, \tau_2 \rightarrow \infty} \left( \int_{-\tau_1}^{z_1} \int_{-\tau_2}^{z_2} d\boldsymbol{\xi} \left( \int_{[-T, T]^2} e^{-i\mathbf{u} \cdot \boldsymbol{\xi}} \left( 1 - \frac{|u_1|}{T} \right) \left( 1 - \frac{|u_2|}{T} \right) (\phi(\mathbf{u}) - \chi(\mathbf{u})) d\mathbf{u} \right) \right). \end{aligned}$$

This shows that the Radon-Nikodym derivative of  $\Delta * \mu_T$  with respect to Lebesgue measure is precisely

$$h_T(\boldsymbol{\xi}) := \frac{T^2}{2\pi} \int_{[-T, T]^2} e^{-i\mathbf{u} \cdot \boldsymbol{\xi}} \left( 1 - \frac{|u_1|}{T} \right) \left( 1 - \frac{|u_2|}{T} \right) (\phi(\mathbf{u}) - \chi(\mathbf{u})) du_1 du_2.$$

For  $\epsilon > 0$ , let  $\mathcal{S}_\epsilon := \{\mathbf{u} \in [-T, T]^2 : \epsilon \leq |u_1|, |u_2| \leq T\}$  and choose a smooth function  $k_\epsilon : \mathbb{R}^2 \rightarrow [0, 1]$  with support lying inside of  $\mathcal{S}_\epsilon$  such that  $k_\epsilon \rightarrow 1$  uniformly on  $[-T, T]^2$  as  $\epsilon \rightarrow 0$ . Define

$$h_{T, \epsilon}(\boldsymbol{\xi}) := \frac{T^2}{2\pi} \int_{[-T, T]^2} e^{-i\mathbf{u} \cdot \boldsymbol{\xi}} \left( 1 - \frac{|u_1|}{T} \right) \left( 1 - \frac{|u_2|}{T} \right) (\phi(\mathbf{u}) - \chi(\mathbf{u})) k_\epsilon(\mathbf{u}) du_1 du_2.$$



By the Dominated Convergence theorem,  $h_{T,\epsilon} \rightarrow h_T$  as  $\epsilon \rightarrow 0$  uniformly on  $[-T, T]^2$ . Now, define the functions

$$J_{\epsilon,1}(\boldsymbol{\xi}) := \frac{T^2}{2\pi} \int_{S_\epsilon} \frac{e^{-i\mathbf{u} \cdot \boldsymbol{\xi}}}{-2iu_1} \left(1 - \frac{|u_1|}{T}\right) \left(1 - \frac{|u_2|}{T}\right) (\phi(\mathbf{u}) - \chi(\mathbf{u})) k_\epsilon(\mathbf{u}) du_1 du_2,$$

$$J_{\epsilon,2}(\boldsymbol{\xi}) := \frac{T^2}{2\pi} \int_{S_\epsilon} \frac{e^{-i\mathbf{u} \cdot \boldsymbol{\xi}}}{2iu_2} \left(1 - \frac{|u_1|}{T}\right) \left(1 - \frac{|u_2|}{T}\right) (\phi(\mathbf{u}) - \chi(\mathbf{u})) k_\epsilon(\mathbf{u}) du_1 du_2.$$

Note that the integrand is differentiable almost everywhere. Differentiating under the integral sign gives

$$h_{T,\epsilon}(\boldsymbol{\xi}) = \frac{\partial J_{\epsilon,1}}{\partial \xi_1}(\boldsymbol{\xi}) - \frac{\partial J_{\epsilon,2}}{\partial \xi_2}(\boldsymbol{\xi}).$$

Fix  $\tau_1, \tau_2 > 0$ , set  $A(\boldsymbol{\tau}, \mathbf{z}) := [-\tau_1, z_1] \times [-\tau_2, z_2]$ , and let  $R = R_{\tau_1, \tau_2}$  denote the counterclockwise oriented rectangle with corners  $(-\tau_1, -\tau_2)$ ,  $(z_1, -\tau_2)$ ,  $(z_1, z_2)$  and  $(-\tau_1, z_2)$ . By Green's theorem,

$$\begin{aligned} \int_{A(\boldsymbol{\tau}, \mathbf{z})} h_{T,\epsilon}(\boldsymbol{\xi}) d\xi_1 d\xi_2 &= \int_R (J_{\epsilon,2}(\boldsymbol{\xi}) d\xi_1 + J_{\epsilon,1}(\boldsymbol{\xi}) d\xi_2) \\ &= \int_{-\tau_1}^{z_1} (J_{\epsilon,2}(\xi_1, -\tau_2) - J_{\epsilon,2}(\xi_1, z_2)) d\xi_1 + \int_{-\tau_2}^{z_2} (J_{\epsilon,1}(-\tau_1, \xi_2) - J_{\epsilon,1}(z_1, \xi_2)) d\xi_2 \\ &= \frac{T^2}{2\pi} \int_{[-T, T]^2} du_1 du_2 \left(1 - \frac{|u_1|}{T}\right) \left(1 - \frac{|u_2|}{T}\right) (\phi(\mathbf{u}) - \chi(\mathbf{u})) k_\epsilon(\mathbf{u}) \\ &\quad \cdot \left( \left( \frac{e^{-iu_2 z_2} - e^{iu_2 \tau_2}}{2i} \right) \int_{-\tau_1}^{z_1} \frac{e^{-iu_1 \xi_1}}{u_2} d\xi_1 - \left( \frac{e^{iu_1 \tau_1} - e^{-iu_1 z_1}}{2i} \right) \int_{-\tau_2}^{z_2} \frac{e^{-iu_2 \xi_2}}{u_1} d\xi_2 \right). \end{aligned}$$

Evaluating the integrals and bounding trivially then yields

$$\begin{aligned} &\left| \int_{-\tau_1}^{z_1} \int_{-\tau_2}^{z_2} h_{T,\epsilon}(\boldsymbol{\xi}) d\xi_1 d\xi_2 \right| \\ &\leq \frac{T^2}{2\pi} du_1 du_2 \int_{[-T, T]^2} du_1 du_2 \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} \\ &\quad \cdot \left| \left( \frac{e^{-iu_2 z_2} - e^{iu_2 \tau_2}}{2} \right) (e^{-iu_1 z_1} - e^{iu_1 \tau_1}) - \left( \frac{e^{iu_1 \tau_1} - e^{-iu_1 z_1}}{2} \right) (e^{-iu_2 z_2} - e^{iu_2 \tau_2}) \right| \\ &\ll \int_{[-T, T]^2} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} |(e^{-iu_1(z_1+\tau_1)} - 1)(e^{-iu_2(z_2+\tau_2)} - 1)| du_1 du_2. \end{aligned}$$

Note that this holds for all  $\epsilon > 0$ , so again by the Dominated Convergence theorem, we also have

$$\left| \int_{-\tau_1}^{z_1} \int_{-\tau_2}^{z_2} h_T(\boldsymbol{\xi}) d\xi_1 d\xi_2 \right| \ll \int_{[-T, T]^2} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} |(e^{-iu_1(z_1+\tau_1)} - 1)(e^{-iu_2(z_2+\tau_2)} - 1)| du_1 du_2.$$

Suppose that  $|\tau_1|, |\tau_2| \leq T$  and fix  $\mathbf{u} \in [-T, T]^2 \setminus \mathcal{R}_T$ , and suppose without loss of generality that  $|u_1| \leq T^{-3}$  (the alternative case being similarly argued). As  $|e^{-iu_1(z_1+\tau_1)} - 1| \leq$

$$2|u_1||z_1 + \tau_1|,$$

$$\begin{aligned} \int_{|u_1| \leq T^{-3}} \int_{|u_2| \in [-T^{-3}, T]} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} |(e^{-iu_1(z_1 + \tau_1)} - 1)(e^{-iu_1(z_1 + \tau_1)} - 1)| du_1 du_2 \\ \leq |z_1 + \tau_1| T^{-2} \ll T^{-1}. \end{aligned}$$

Similarly, if both  $|u_1|, |u_2| \leq T^{-3}$ , we have

$$\int_{[-T^{-3}, T^{-3}]} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} |(e^{-iu_1(z_1 + \tau_1)} - 1)(e^{-iu_1(z_1 + \tau_1)} - 1)| du_1 du_2 \leq |z_1 + \tau_1| |z_2 + \tau_2| T^{-6} \ll T^{-1}.$$

Thus, we have

$$\int_{[-T, T]^2 \setminus \mathcal{R}_T} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} |(e^{-iu_1(z_1 + \tau_1)} - 1)(e^{-iu_1(z_1 + \tau_1)} - 1)| du_1 du_2 \ll T^{-1},$$

and hence, choosing  $\boldsymbol{\tau} = (T, T)$ ,

$$\begin{aligned} |\Delta * \mu_T(A((T, T), \mathbf{z}))| &\ll \int_{\mathcal{R}_T} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} |(e^{-iu_1(z_1 + \tau_1)} - 1)(e^{-iu_1(z_1 + \tau_1)} - 1)| du_1 du_2 + T^{-1} \\ &\ll \int_{\mathcal{R}_T} \frac{|\phi(\mathbf{u}) - \chi(\mathbf{u})|}{|u_1 u_2|} du_1 du_2 + T^{-1}. \end{aligned}$$

Since the convolution operator  $\Delta \mapsto \Delta * \mu_T$  has norm  $\|W_T\|_{L^1(\mathbb{R}^2)} \leq T^2$  on the space of probability measures, (14) implies that

$$\begin{aligned} |\Delta * \mu_T(B(\mathbf{z}) \setminus A((T, T), \mathbf{z}))| &\leq T^2 |\Delta(B(\mathbf{z}) \setminus A((T, T), \mathbf{z}))| \\ &= T^2 |G(\mathbf{z}) - H(\mathbf{z}) - \int_{-T}^{z_1} \int_{-T}^{z_2} d(G - H)(\mathbf{u})| \ll T. \end{aligned}$$

Combining this with (17) and dividing by  $T^2$  proves the claim.  $\square$

We next show that Lemma 3.1 is applicable to the case that  $H = F_{x, S, a}$  and  $G = \Phi_{(2)}$  when  $S$  is siftable with respect to  $a$ .

**Lemma 3.2.** *Let  $a \in \mathbb{N}$  and let  $S$  be siftable with respect to  $a$ . Then for  $\epsilon_1, \epsilon_2 \in \{0, 1\}$ ,*

$$\sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) \left( \frac{\omega(n) - \lambda(x)}{\sqrt{\lambda(x)}} \right)^{2\epsilon_1} \left( \frac{\omega(n+a) - \lambda(x)}{\sqrt{\lambda(x)}} \right)^{2\epsilon_2} = E(x; a) \left( 1 + O \left( \frac{1}{\sqrt{\lambda(x)}} \right) \right).$$

*Proof.* We will only prove the case that  $\epsilon_1 = \epsilon_2 = 1$ , as the remaining cases are either trivial or follow from a simpler version of the argument we present here.

We follow the method of Granville and Soundararajan [11]. For each prime  $p \leq x$ , define  $g_p(n) := 1 - f(p)$  when  $p|n$ , and  $g_p(n) = -f(p)$  when  $p \nmid n$ . Let  $z = x^{\frac{\theta}{4}}$ . For any  $m \leq x$ ,

$$\omega(m) - \lambda(x) = \sum_{p|m} (1 - f(p)) - \sum_{\substack{p \leq x \\ p \nmid m}} f(p) = \sum_{p \leq x} g_p(m) = \sum_{p \leq y} g_p(m) + O(1).$$

Moreover, for  $m = \prod_j p_j^{\alpha_j}$  put  $g_m(n) := \prod_j g_{p_j}(n)^{\alpha_j}$ . Then

$$\begin{aligned} \mu_{2,2}(x) &:= \sum_{\substack{n \leq x \\ (n,a)=1}} 1_S(n) 1_S(n+a) (\omega(n) - \lambda(x))^2 (\omega(n+a) - \lambda(x))^2 \\ &= \sum_{p_1, p_2, q_1, q_2 \leq z} \sum_{\substack{n \leq x \\ (n,a)=1}} 1_S(n) 1_S(n+a) g_{p_1 p_2}(n) g_{q_1 q_2}(n+a) \\ &\quad + O \left( \sum_{\substack{n \leq x \\ (n,a)=1}} 1_S(n) 1_S(n+a) \left| \sum_{p \leq z} g_p(n) \right| \left| \sum_{q \leq z} g_q(n+a) \right| \left( \left| \sum_{p \leq z} g_p(n) \right| + \left| \sum_{q \leq z} g_q(n+a) \right| \right) \right). \end{aligned}$$

Let  $M_{2,2}(x; p_1, p_2, q_1, q_2)$  be the first term above. Since  $g_{p_1 p_2}(n) = g_{p_1 p_2}((n, p_1 p_2))$ , we have

$$\begin{aligned} M_{2,2}(x; p_1, p_2, q_1, q_2) &= \sum_{d|p_1 p_2} \sum_{e|q_1 q_2} g_{p_1 p_2}(d) g_{q_1 q_2}(e) \sum_{\substack{n \leq x \\ (n, p_1 p_2)=d, (n+a, q_1 q_2)=e}} 1_S(n) 1_S(n+a) 1_{(n,a)=1} \\ &= \sum_{d|p_1 p_2} \sum_{e|q_1 q_2} g_{p_1 p_2}(d) g_{q_1 q_2}(e) \sum_{d'|p_1 p_2/d} \mu(d') \sum_{e'|q_1 q_2/e} \mu(e') \sum_{\substack{n \leq x \\ n \equiv 0(dd'), n+a \equiv 0(ee')}} 1_S(n) 1_S(n+a) 1_{(n,a)=1}. \end{aligned}$$

Note that  $(dd', ee') = 1$  since  $(n, a) = 1$  for each  $n$  in the support of the sum. Since  $S$  is siftable with respect to  $a$ , we have

$$\begin{aligned} M_{2,2}(x; p_1, p_2, q_1, q_2) &= \sum_{d|p_1 p_2} \sum_{e|q_1 q_2} g_{p_1 p_2}(d) g_{q_1 q_2}(e) \\ &\quad \cdot \sum_{d'|p_1 p_2/d} \mu(d') \sum_{e'|q_1 q_2/e} \mu(e') f(dd' ee') (E(x; a) + R(x; dd', ee')) \\ &= E(x; a) \left( \sum_{dd'|p_1 p_2} \mu(d') g_{p_1 p_2}(d) f(dd') \right) \left( \sum_{ee'|p_1 p_2} \mu(e') g_{q_1 q_2}(e) f(ee') \right) \\ &\quad + \sum_{dd'|p_1 p_2} \sum_{ee'|q_1 q_2} \mu(d' e') g_{p_1 p_2}(d) g_{q_1 q_2}(e) f(dd' ee') R(x; dd', ee'). \end{aligned}$$

Since the terms depending on  $p_1 p_2$  are identical (up to relabelling) to those depending on  $q_1 q_2$ , we shall only write out the treatment of the former terms. We first consider the case that  $p_1 \neq p_2$ . In that case,  $(d, d') = 1$  necessarily, and thus

$$\begin{aligned} \sum_{dd'|p_1 p_2} \mu(d') g_{p_1 p_2}(d) f(dd') &= \sum_{d|p_1 p_2} g_{p_1 p_2}(d) f(d) \prod_{p|p_1 p_2/d} (1 - f(p)) \\ &= \left( \prod_{p|p_1 p_2} (1 - f(p)) \right) \sum_{d|p_1 p_2} g_{p_1 p_2}(d) \prod_{p|d} f(p) (1 - f(p))^{-1} \\ &= \left( \prod_{p|p_1 p_2} (1 - f(p)) \right) (f(p_1) f(p_2) - f(p_1) f(p_2) - f(p_1) f(p_2) + f(p_1) f(p_2)) = 0. \end{aligned}$$

It follows that the off-diagonal terms contribute

$$\begin{aligned} & \sum_{\substack{p_1, p_2, q_1, q_2 \leq y \\ p_1 \neq p_2 \text{ or } q_1 \neq q_2}} M_{2,2}(x; p_1, p_2, q_1, q_2) \\ &= \sum_{\substack{p_1, p_2, q_1, q_2 \leq z \\ p_1 \neq p_2 \text{ or } q_1 \neq q_2}} \sum_{dd' | p_1 p_2} \sum_{ee' | q_1 q_2} \mu(d' e') g_{p_1 p_2}(d) g_{q_1 q_2}(e) f(dd' ee') R(x; dd', ee') =: \mathcal{R}_1. \end{aligned}$$

Conversely, if  $p_1 = p_2$  then a simple computation shows that

$$\begin{aligned} \sum_{dd' | p_1 p_2} \mu(d') g_{p_1 p_2}(d) f(dd') &= f(p_1)^2 + f(p_1)(1 - f(p_1))^2 - f(p_1)^3 \\ &= f(p_1) + O(f(p_1)^2). \end{aligned}$$

The diagonal terms thus contribute

$$\begin{aligned} \sum_{p, q \leq z} M_{2,2}(x; p, p, q, q) &= E(x; a) \left( \sum_{p \leq z} f(p) + O(1) \right)^2 \\ &\quad + \sum_{p, q \leq z} \sum_{dd' | p^2} \sum_{ee' | q^2} \mu(d' e') g_{p^2}(d) g_{q^2}(e) f(dd' ee') R(x; dd', ee') \\ &= E(x; a) (\lambda(x) + O(1))^2 + \mathcal{R}_2. \end{aligned}$$

Now, for any  $m \in \mathbb{N}$ ,  $|g_m| \leq 1$ , so that since  $\beta \rightarrow \infty$ ,

$$\mathcal{R}_1 \ll \sum_{p_1, p_2, q_1, q_2 \leq z} \max_{\substack{dd' | p_1 p_2 \\ ee' | q_1 q_2}} |R(x; dd', ee')| \leq \sum_{m_1 m_2 \leq z^4} |R(x; m_1, m_2)| \ll x(\log x)^{-3}.$$

We can estimate  $\mathcal{R}_2$  the same way. It follows that

$$\begin{aligned} \mu_{2,2}(x) &= E(x; a) \lambda(x)^2 \left( 1 + O\left(\frac{1}{\lambda(x)}\right) \right) \\ &\quad + O\left( \sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) \left| \sum_{p \leq z} g_p(n) \right| \left| \sum_{q \leq z} g_q(n+a) \right| \left( \left| \sum_{p \leq z} g_p(n) \right| + \left| \sum_{q \leq z} g_q(n+a) \right| \right) \right). \end{aligned}$$

We will estimate

$$\sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) \left| \sum_{p \leq z} g_p(n) \right|^2 \left| \sum_{q \leq z} g_q(n+a) \right|,$$

the remaining term being estimated in exactly the same way. By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \sum_{\substack{n \leq x \\ (n,a)=1}} 1_S(n) 1_S(n+a) \left| \sum_{p \leq z} g_p(n) \right|^2 \left| \sum_{q \leq z} g_q(n+a) \right|^2 \\ & \leq \left( \sum_{\substack{n \leq x \\ (n,a)=1}} 1_S(n) 1_S(n+a) \left( \sum_{p \leq z} g_p(n) \right)^2 \left( \sum_{q \leq z} g_q(n+a) \right)^2 \right)^{\frac{1}{2}} \\ & \quad \cdot \left( \sum_{\substack{n \leq x \\ (n,a)=1}} 1_S(n) 1_S(n+a) \left( \sum_{p \leq z} g_p(n) \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The first sum here is precisely the  $(2,2)$ -moment we just estimated, and its square root is thus  $\ll \lambda(x) \sqrt{E(x;a)}$ . The second is the  $(2,0)$ -moment, i.e., where  $\epsilon_1 = 1$  and  $\epsilon_2 = 0$ , and its square root is  $\ll \sqrt{\lambda(x)E(x;a)}$ . Hence,

$$\mu_{2,2}(x) = E(x;a) \lambda(x)^2 \left( 1 + O \left( \frac{1}{\sqrt{\lambda(x)}} \right) \right).$$

Dividing both sides by  $\lambda(x)^2$  now suffices to prove the claim.  $\square$

**Corollary 3.3.** *If  $a \in \mathbb{N}$  and  $S$  is siftable with respect to  $a$  then for each fixed  $\mathbf{z}$  and  $T \geq 1$ ,*

$$\left| F_{x,S,a}(\mathbf{z}) - \Phi_{(2)}(\mathbf{z}) - \int_{-T}^{z_1} \int_{-T}^{z_2} d(F_{x,S,a} - \Phi_{(2)})(\mathbf{u}) \right| \ll T^{-1}.$$

*Proof.* Observe that

$$\left| \Phi_{(2)}(\mathbf{z}) - \int_{-T}^{z_1} \int_{-T}^{z_2} d\Phi_{(2)}(\mathbf{w}) \right| \ll \int_{-\infty}^{-T} \int_{\mathbb{R}} e^{-\frac{1}{2}(u^2+v^2)} dudv \ll e^{-\frac{1}{4}T^2} \ll T^{-1}.$$

Moreover, suppose that  $n \leq x$  is counted by  $F_{x,S,a}(\mathbf{z}) - \int_{-T}^{z_1} \int_{-T}^{z_2} dF_{x,S,a}(\mathbf{w})$ . Then  $\min\{\tilde{\omega}_x(n), \tilde{\omega}_x(n+a)\} < -T$ . It follows by Lemma 3.2 that

$$\begin{aligned} & \left| F_{x,S,a}(\mathbf{z}) - \int_{-T}^{z_1} \int_{-T}^{z_2} dF_{x,S,a}(\mathbf{w}) \right| \\ & \ll \sum_{\epsilon \in \{0,1\}} E(x;a)^{-1} |\{n \leq x : n, n+a \in S, (n,a)=1, \left| \frac{\omega(n+\epsilon a) - \lambda(x)}{\sqrt{\lambda(x)}} \right| > T\}| \\ & \ll T^{-2} E(x;a)^{-1} \sum_{\substack{n \leq x \\ (a,n)=1}} 1_S(n) 1_S(n+a) \left( 1 + \left( \frac{\omega(n) - \lambda(x)}{\sqrt{\lambda(x)}} \right)^2 \right) \left( 1 + \left( \frac{\omega(n+a) - \lambda(x)}{\sqrt{\lambda(x)}} \right)^2 \right) \\ & \ll T^{-2}. \end{aligned}$$

This more than suffices to prove the claim.  $\square$

The following simple lemma allows us to put into effect the Poisson approximation results of Section 3.

**Lemma 3.4.** *Let  $W, Z$  be random vectors on a common probability space taking values in  $\mathbb{R}^n$  with respective laws  $\mu$  and  $\nu$ . Then*

$$\sup_{\mathbf{u} \in \mathbb{R}^n} |\mathbb{E}(e^{i\mathbf{u} \cdot W}) - \mathbb{E}(e^{i\mathbf{u} \cdot Z})| \ll d_{TV}(\mu, \nu).$$

*Proof.* Fix  $\mathbf{u} \in \mathbb{R}^n$ . For each  $r \in [0, 2\pi)$  let  $A(r) := \{\mathbf{t} \in \mathbb{R}^n : \mathbf{t} \cdot \mathbf{u} \equiv r(2\pi)\}$ , which is clearly Borel measurable. Then

$$\begin{aligned} |\mathbb{E}(e^{i\mathbf{u} \cdot W}) - \mathbb{E}(e^{i\mathbf{u} \cdot Z})| &= \left| \int_0^{2\pi} dr e^{ir} \int_{A(r)} d(\mu - \nu)(\mathbf{t}) \right| \leq 2\pi \sup_{r \in [0, 2\pi)} |\mu(A(r)) - \nu(A(r))| \\ &\ll d_{TV}(\mu, \nu), \end{aligned}$$

as claimed.  $\square$

The following is a trivial calculation.

**Lemma 3.5.** *Let  $X$  be a Poisson random variable with parameter  $\lambda$ , and let  $\tilde{X} := (X - \lambda)/\sqrt{\lambda}$ . Let  $\pi$  denote the characteristic function of  $\tilde{X}$ . Then*

$$|\pi(w)| = \exp\left(-\lambda(1 - \cos(w/\sqrt{\lambda}))\right).$$

We thus have the following.

**Proposition 3.6.** *Let  $a \in \mathbb{N}$  and let  $S$  be a siftable, regular set (with respect to  $a$ ). Let  $|u|, |v| \leq 2\pi\sqrt{\log_2 x}$ , and let  $w := \max\{|u|, |v|\}$ . Then*

$$(18) \quad |\phi_{x,y}(u, v) - \chi(u)\chi(v)| \ll \frac{1}{\log_2 y} + R(x; y)$$

$$(19) \quad |\phi_x(u, v) - \chi(u)\chi(v)| \ll \frac{w \log_2(\mathfrak{L}/w)}{(\log_2 x)^{\frac{1}{2}}}.$$

*Proof.* Let  $Z := (Z_1, Z_2) \sim \text{Poi}(\lambda(y))^2$  be the independent random variables from Lemma 2.8. Let  $\tilde{Z}_j := (Z_j - \lambda(y))/\sqrt{\lambda(y)}$ ,  $\tilde{Z} := (\tilde{Z}_1, \tilde{Z}_2)$  and let  $\pi_y$  denote the characteristic function of  $\tilde{Z}_j$ . Since  $S$  is regular, we note that  $\lambda(y) \gg \log_2 y$ . Note that by independence, we have  $\mathbb{E}(e^{i(u,v) \cdot \tilde{Z}}) = \pi_y(u)\pi_y(v)$ . By Lemmata 2.8 and 3.4, we have

$$(20) \quad |\phi_{x,y}(u, v) - \pi_y(u)\pi_y(v)| \ll d_{TV}(\nu_y, \mathcal{L}(\tilde{Z})) \ll \frac{1}{\log_2 y} + R(x; y).$$

We now compare  $\pi_y(u)\pi_y(v)$  to  $\chi(u)\chi(v)$ . We note that when  $|v^4/\lambda(y)| < 1$  we have

$$|\pi_y(v) - \chi(v)| = \left| \exp\left(-\lambda(y)(1 - \cos(v/\sqrt{\lambda(y)}))\right) - e^{-\frac{1}{2}v^2} \right| \ll e^{-\frac{1}{2}v^2} v^4 \lambda(y)^{-1}.$$

On the other hand, when  $|v^4/\lambda(y)| > 1$  then

$$|\pi_y(v) - \chi(v)| \leq |\pi_y(v)| + e^{-\frac{1}{2}v^2} \ll e^{-\frac{1}{3}\sqrt{\lambda(y)}}.$$

It follows that

$$\begin{aligned} |\pi_y(u)\pi_y(v) - \chi(u)\chi(v)| &\ll \max\{|\pi_y(v) - \chi(v)|, |\pi_y(u) - \chi(u)|\} \\ &\ll e^{-\frac{1}{2}w^2} w^4 \lambda(y)^{-1} 1_{|w| < \lambda(y)^{1/4}} + e^{-\frac{1}{3}\sqrt{\lambda(y)}} 1_{|w| > \lambda(y)^{1/4}}. \end{aligned}$$

This, coupled with (20), implies (18).

Now, we consider the difference between  $\phi_x$  and  $\phi_{x,y}$ . We have

$$\begin{aligned}
|\phi_x(u, v) - \phi_{x,y}(u, v)| &\leq E(x; a)^{-1} \sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) |e^{iu(\tilde{\omega}_x(n) - \tilde{\omega}_y(n))} e^{iv(\tilde{\omega}_x(n+a) - \tilde{\omega}_y(n+a))} - 1| \\
&\ll E(x; a)^{-1} \sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) \\
&\quad \cdot \max\{|u| |\tilde{\omega}_x(n) - \tilde{\omega}_y(n)|, |v| |\tilde{\omega}_x(n+a) - \tilde{\omega}_y(n+a)|\} \\
&\ll \max\{|u|, |v|\} E(x; a)^{-1} \sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) \\
&\quad \cdot \left( \frac{|\omega(n) - \omega_y(n)| - |\lambda(x) - \lambda(y)|}{\sqrt{\lambda(y)}} + \left( 1 - \sqrt{\frac{\lambda(y)}{\lambda(x)}} \right) |\omega(n) - \lambda(x)| \right) \\
&\ll w \left( \frac{\beta}{\sqrt{\log_2 x}} + \left( \frac{\log \beta}{\log_2 x} \right) E(x; a)^{-1} \sum_{\substack{n \leq x \\ (n, a) = 1}} 1_S(n) 1_S(n+a) |\omega(n) - \lambda(x)| \right).
\end{aligned}$$

By the Cauchy-Schwarz inequality and Lemma 3.2, we have

$$E(x; a)^{-1} \sum_{n \leq x} 1_S(n) 1_S(n+a) 1_{(n, a) = 1} |\omega(n) - \lambda(x)| \ll \sqrt{\lambda(x)},$$

so that  $|\phi_x(u, v) - \phi_{x,y}(u, v)| \ll w \beta (\log_2 x)^{-\frac{1}{2}}$ . Choose  $y$  such that  $\beta \log \beta = 21 \log(\mathfrak{L}/w)$ . Then, upon applying (18),

$$\begin{aligned}
|\phi_x(u, v) - \chi(u)\chi(v)| &\leq |\phi_x(u, v) - \phi_{x,y}(u, v)| + |\phi_{x,y}(u, v) - \chi(u)\chi(v)| \\
&\ll \frac{w \log_2(\mathfrak{L}/w)}{(\log_2 x)^{\frac{1}{2}}} + \frac{1}{\log(\log x \log(\mathfrak{L}/w))} + e^{-\frac{1}{2}w^2} w^4 (\log(\log x \log(\mathfrak{L}/w)))^{-1},
\end{aligned}$$

which completes the proof of (19).  $\square$

*Proof of Theorem 2.3.* Let  $T := (\log_2 x)^{\frac{1}{4}} (\log_3 x)^{-1}$ . By Lemma 3.1, we have

$$\|F_x - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} \ll \int_{\mathcal{R}_T} \frac{|\phi_x(u, v) - \chi(u)\chi(v)|}{|uv|} dudv + T^{-1},$$

where we recall that  $\mathcal{R}_T := \{\mathbf{u} \in [-T, T]^2 : |u_1|, |u_2| \geq T^{-3}\}$ . From (19), we have  $|\phi_x(u, v) - \chi(u)\chi(v)| \ll w \frac{\log_3 x}{\sqrt{\log_2 x}}$  for  $w := \max\{|u|, |v|\}$ . Thus,

$$\begin{aligned}
\int_{\mathcal{R}_T} \frac{|\phi_x(u, v) - \chi(u)\chi(v)|}{|uv|} dudv &\leq 2 \int_{|u| \in [T^{-3}, T]} \int_{|v| \leq |u|} \frac{|\phi_x(u, v) - \chi(u)\chi(v)|}{|uv|} dudv \\
&\ll \frac{\log_3 x}{\sqrt{\log_2 x}} \int_{|u| \in [T^{-3}, T]} du \int_{|v| \in [T^{-3}, T]} \frac{dv}{v} \ll \frac{\log_3 x}{\sqrt{\log_2 x}} (T \log T) = (\log_3 x)(\log_2 x)^{-\frac{1}{4}}.
\end{aligned}$$

Thus, we get  $\|F_x - \Phi^2\|_{L^\infty(\mathbb{R}^2)} \ll (\log_3 x)(\log_2 x)^{-\frac{1}{4}}$ , and the proof of Theorem 2.3 is complete in the case that  $1_S$  is not multiplicative.

When  $1_S$  is multiplicative and  $S$  is siftable for each  $\alpha|a$  then as  $1_S(n)1_S(n+a) = 1_S(m)1_S(m+\gamma)$ , where  $\alpha\gamma = a$  and  $m\alpha = n$ ,

$$\begin{aligned}\phi_x^*(u, v) &= E^*(x; a)^{-1} \sum_{\alpha\gamma=a} e^{i(u+v)\lambda(x)^{-\frac{1}{2}}\omega(\alpha)} \sum_{\substack{m \leq x/\alpha \\ (m, \gamma)=1}} 1_S(m)1_S(m+\gamma) e^{i(u\tilde{\omega}_x(m)+v\tilde{\omega}_x(m+\gamma))} \\ &= \sum_{\alpha\gamma=a} e^{i(u+v)\lambda(x)^{-\frac{1}{2}}\omega(\alpha)} \frac{E(x/\alpha; \gamma)}{E^*(x; a)} \phi_{x/\alpha, S, \gamma}(u, v).\end{aligned}$$

The result now follows by applying the arguments for the previous case to the average pointwise distance  $|\phi_{x/\alpha, S, \gamma}(u, v) - \chi(u, v)|$  for each  $\alpha|a$ .  $\square$

#### 4. A SIEVE RESULT FOR CONSECUTIVE SQUAREFREE INTEGERS

Throughout this section, let  $E(x) := x \prod_p (1 - 2p^{-2})$  and for  $q \geq 2$ , let  $E_q(x) := E(x) \prod_{p|q} (1 - 2p^{-2})^{-1}$ . It is well-known that  $E(x)$  is asymptotically the number of integers  $n \leq x$  such that  $n$  and  $n+1$  are both squarefree (see, for instance, [12]). We show here that  $S$  is siftable and regular with respect to each fixed  $a \in \mathbb{N}$ , in the sense of Definition 2.1.

Since property ii) in Definition 2.1 requires information about the distribution of squarefree integers in certain arithmetic progressions, we begin with the following.

**Lemma 4.1.** *Let  $q \geq 2$  and let  $c$  be a reduced residue class modulo  $q$ , and suppose that  $(q, c)$  is squarefree. Let  $a \geq 1$ . Then*

$$\sum_{\substack{n \leq x \\ n \equiv c(q)}} \mu^2(n) \mu^2(n+a) = \frac{E_q(x)}{q} \prod_{p|(q, c(c+a))} \left(1 - \frac{1}{p}\right) \prod_{p^2|(q, c+a)} \left(1 - \frac{1}{p^2}\right) \prod_{p^2|a} \left(1 + \frac{1}{p^2}\right) + O\left(x^{\frac{2}{3}+\epsilon} \tau(q)\right).$$

The restriction that  $(q, c)$  be squarefree makes the result non-trivial, as if  $q$  and  $c$  shared a common factor  $b^2$  say, then if  $n \equiv c(q)$  then  $b^2|n$ . Thus, the sum here would be 0.

*Proof.* For  $e_1, e_2 \leq x^{\frac{1}{2}}$  let

$$N_a(x; e_1, e_2, c, q) := \left| \{n \leq x : e_1^2|n, e_2^2|(n+a), n \equiv c(q)\} \right|.$$

Note that  $N_a(x; e_1, e_2, c, q)$  is zero as long as  $(e_1, e_2)^2 \nmid a$ . Let  $2 \leq z \leq x^{\frac{1}{2}}$  be a parameter to be chosen. Then

$$\begin{aligned}\sum_{\substack{n \leq x \\ n \equiv c(q)}} \mu^2(n) \mu^2(n+a) &= \sum_{\substack{e_1, e_2 \leq x^{\frac{1}{2}} \\ (e_1, e_2)^2|a}} \mu(e_1) \mu(e_2) N_a(x; e_1, e_2, c, q) \\ &= \sum_{\substack{e_1 e_2 \leq z \\ (e_1, e_2)^2|a}} \mu(e_1) \mu(e_2) N_a(x; e_1, e_2, c, q) + \sum_{\substack{e_1, e_2 \leq x^{\frac{1}{2}} \\ e_1 e_2 > z, (e_1, e_2)^2|a}} \mu(e_1) \mu(e_2) N_a(x; e_1, e_2, c, q) \\ &=: T_1 + T_2.\end{aligned}$$

Observe that  $N_a(x; e_1, e_2, c, q)$  counts simultaneous solutions to the triple of congruences  $n \equiv c(q)$ ,  $n \equiv 0(e_1^2)$  and  $n \equiv -a(e_2^2)$ . By the Chinese remainder theorem, a solution



exists modulo  $[q, e_1^2, e_2^2]$  if, and only if,  $(q, e_1^2)|c$  and  $(q, e_2^2)|(c+a)$  (and  $(e_1, e_2)^2|a$ , as we are already assuming). As such, we have

$$\begin{aligned}
T_1 &= x \sum_{\substack{e_1 e_2 \leq z \\ (e_1, e_2)^2 | a, e_1^2 | c, e_2^2 | (c+a)}} \frac{\mu(e_1)\mu(e_2)}{[q, e_1^2, e_2^2]} + O\left(\sum_{e_1 e_2 \leq z} 1\right) \\
&= \frac{x}{q} \sum_{\substack{e_1 e_2 \leq z \\ (e_1, e_2)^2 | a, (q, e_1^2) | c, (q, e_2^2) | (c+a)}} \frac{\mu(e_1)\mu(e_2)}{(e_1 e_2)^2} (e_1, e_2)^2 \left(q, \left(\frac{e_1 e_2}{(e_1, e_2)}\right)^2\right) + O\left(\sum_{m \leq z} \tau(m)\right) \\
&= \frac{x}{q} \sum_{\delta^2 | a} \delta^2 \sum_{\substack{e_1 e_2 \leq y \\ (e_1, e_2) = \delta, (q, e_1^2) | c, (q, e_2^2) | (c+a)}} \frac{\mu(e_1)\mu(e_2)}{(e_1 e_2)^2} (q, e_1^2)(q, (e_2/\delta)^2) + O(z \log z) \\
&= \frac{x}{q} \sum_{\delta^2 | a} \delta^2 \sum_{f_1 | (q, c), f_2 | (q, c+a)} f_1 f_2 \sum_{\substack{e_1 e_2 \leq z \\ (e_1, e_2) = \delta, (e_1^2, q) = f_1, ((e_2/\delta)^2, q) = f_2}} \frac{\mu(e_1)\mu(e_2)}{(e_1 e_2)^2} + O(z \log z) \\
&= \frac{x}{q} \sum_{\delta^2 | a} \delta^2 \sum_{f_1 | (q, c), f_2 | (q, c+a)} f_1 f_2 \sum_{\substack{(e_1, e_2) = \delta \\ ((e_1/\delta)^2, q) = f_1, ((e_2/\delta)^2, q) = f_2}} \frac{\mu(e_1)\mu(e_2)}{(e_1 e_2)^2} + O\left(z \log z + \frac{x}{qz} \tau(q)\right).
\end{aligned}$$

Decompose  $(q, c+a) = uv$ , where  $(u, v) = 1$ ,  $\mu^2(v) = 1$  and  $u$  is squarefull. As such, we can split  $f_2 = gh$  where  $g := (f_2, u)$  and  $h := (f_2, v)$ . Further, write  $\tilde{e}_2 := e_2/\delta$ . Note that if  $(\tilde{e}_2^2, q)|c$  then  $(\tilde{e}_2^2, (q, c)) = (\tilde{e}_2, q)$  (since  $(q, c)$  is squarefree), and similarly  $(\tilde{e}_2^2, q) = (\tilde{e}_2^2, (q, c+a))$ . In particular,  $(\tilde{e}_2^2, u) = g$  and  $(\tilde{e}_2^2, v) = (e_2, v) = h$ . Also,  $\mu(\tilde{e}_2) \neq 0$  if, and only if,  $g$  is the square of a squarefree integer. Thus, define  $G$  implicitly by  $G^2 = g$ . Then

$$T_1 = \frac{x}{q} \sum_{\delta^2 | a} \delta^2 \sum_{\substack{G^2 | u \\ \mu^2(G) = 1}} \frac{\mu(G)}{G^2} \sum_{h_1 | f_1, h_2 | v} \frac{\mu(h_1 h_2)}{h_1 h_2} \sum_{\substack{(e_1, q) = 1 \\ (e_1, e_2) = \delta}} \sum_{(\tilde{e}_2, q) = 1} \frac{\mu(e_1)\mu(e_2)}{(e_1 e_2)^2} + O\left(z \log z + \frac{x}{qz} \tau(q)\right).$$

Let  $\Sigma_\delta$  denote the double sum in  $e_1$  and  $e_2$  with  $(e_1, e_2) = \delta$ . Note that if  $\mu^2(\delta) = 0$  then  $\Sigma_\delta = 0$ , so assume otherwise. We have

$$\begin{aligned}
\Sigma_\delta &= \sum_{(e_1/\delta, q) = 1} \frac{\mu(e_1)}{e_1^2} \sum_{(e_2/\delta, q) = 1, (e_2, e_1) = \delta} \frac{\mu(e_2)}{e_2^2} = \frac{1}{\delta^4} \sum_{(e_1, q) = 1} \frac{\mu(e_1)}{e_1^2} \sum_{(e_2, qe_1) = 1} \frac{\mu(e_2)}{e_2^2} \\
&= \frac{1}{\delta^4} \sum_{(e_1, q) = 1} \frac{\mu(e_1)}{e_1^2} \prod_{p \nmid e_1 q} \left(1 - \frac{1}{p^2}\right) \\
&= \frac{1}{\delta^4} \prod_{p \nmid q} \left(1 - \frac{1}{p^2}\right) \sum_{(e_1, q) = 1} \frac{\mu(e_1)}{e_1^2} \prod_{p | e_1} \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{1}{\delta^4} \prod_{p \nmid q} \left(1 - \frac{1}{p^2}\right) \prod_{p \nmid q} \left(1 - \frac{1}{p^2 - 1}\right) \\
&= \frac{1}{\delta^4} \prod_{p \nmid q} \left(1 - \frac{2}{p^2}\right).
\end{aligned}$$

Inserting this back into our estimate for  $T_1$  gives

$$\begin{aligned}
T_1 &= \frac{x}{q} \prod_{p \nmid q} \left(1 - \frac{2}{p^2}\right) \sum_{\substack{\delta^2 | a \\ \mu^2(\delta)=1}} \frac{1}{\delta^2} \sum_{\substack{G^2 | u \\ \mu^2(G)=1}} \frac{\mu(G)}{G^2} \sum_{h_1 | f_1, h_2 | v} \frac{\mu(h_1 h_2)}{h_1 h_2} + O\left(z \log z + \frac{x}{qz} \tau(q)\right) \\
&= \frac{x}{q} \prod_{p \nmid q} \left(1 - \frac{2}{p^2}\right) \prod_{p || (q, [c, c+a])} \left(1 - \frac{1}{p}\right) \prod_{p^2 || (q, c+a)} \left(1 - \frac{1}{p^2}\right) \prod_{p^2 || a} \left(1 + \frac{1}{p^2}\right) \\
&\quad + O\left(z \log z + \frac{x}{qz} \tau(q)\right).
\end{aligned}$$

We next estimate  $T_2$ . Decomposing the ranges of  $e_1$  and  $e_2$  dyadically, we have

$$T_2 \ll \sum_{\substack{k, l \leq \frac{\log x}{2 \log 2} \\ k+l \geq \log z}} \sum_{\substack{2^k < e_1 \leq 2^{k+1} \\ 2^l < e_2 \leq 2^{l+1}}} N_a(x; e_1, e_2, c, q) =: \sum_{\substack{k, l \leq \frac{\log x}{2 \log 2} \\ k+l \geq \log z}} M_{k,l}(x) \ll M_{K,L}(x) \log^2 x$$

where  $K$  and  $L$  are the respective values of  $k$  and  $l$  that maximize  $M_{k,l}(x)$ . Assume without loss of generality that  $K \geq L$ , the alternative case being similar. Then

$$M_{K,L}(x) \leq \sum_{2^L < e_2 \leq 2^{L+1}} \sum_{m \leq x/2^{2K}} \sum_{\substack{2^K < e_1 \leq 2^{K+1} \\ me_1^2 \equiv -a(e_2^2), me_1^2 \equiv c(q)}} 1.$$

The number of solutions in  $e_1$  to the simultaneous congruence conditions is at most  $\tau(e_2)\tau(q)$ , since there are at most 2 solutions to these same congruences modulo each prime  $p$  dividing  $[e_2, q]$ , and each such solution lifts uniquely to a solution mod  $p^2$  by Hensel's lemma. Thus, we have

$$\begin{aligned}
M_{K,L}(x) &\leq x 2^{-2K} \sum_{l || (q, c+a)} \sum_{\substack{2^L < e_2 \leq 2^{L+1} \\ (e_2^2, q) = l}} (1 + 2^{K-2L} l q^{-1}) \tau(e_2) \ll x L 2^{L-2K} (1 + \tau(q) 2^{K-2L} q^{-1}) \\
&\ll x \log x (2^{-K} + \tau(q) 2^{-(K+L)} q^{-1}) \ll x z^{-\frac{1}{2}} \log x,
\end{aligned}$$

since  $K \geq \frac{1}{2} \log z$ . As such, we have  $T_2 \ll x z^{-\frac{1}{2}} \log^3 x$ , and

$$\begin{aligned}
\sum_{\substack{n \leq x \\ n \equiv c(q)}} \mu^2(n) \mu^2(n+a) &= \frac{x}{q} \prod_{p \nmid q} \left(1 - \frac{2}{p^2}\right) \prod_{p || (q, [c, c+a])} \left(1 - \frac{1}{p}\right) \prod_{p^2 || (q, c+a)} \left(1 - \frac{1}{p^2}\right) \prod_{p^2 || a} \left(1 + \frac{1}{p^2}\right) \\
&\quad + O\left(z \log z + \frac{x}{\sqrt{z}} \log^3 x + \frac{x}{qz} \tau((q, c(c+a)))\right).
\end{aligned}$$

Choosing  $z = x^{\frac{2}{3}}$  furnishes a bound stronger than the claim.  $\square$

**Lemma 4.2.** *Let  $a \geq 1$ . Let  $q, r \geq 2$  be squarefree, such that  $(q, r) | a$ . Then*

$$\begin{aligned}
(21) \quad \sum_{\substack{n \leq x \\ n \equiv 0(q), n+a \equiv 0(r)}} \mu^2(n) \mu^2(n+a) &= x \frac{\phi([q, r])}{[q, r]^2} \prod_{p^2 || a} \left(1 + \frac{1}{p^2}\right) \prod_{p \nmid [q, r]} \left(1 - \frac{2}{p^2}\right) + O\left(x^{\frac{2}{3}+\epsilon} \tau([q, r])\right).
\end{aligned}$$

Consequently, if  $(k, r) = 1$  then  
(22)

$$\sum_{\substack{n \leq x \\ n \equiv 0(q), n+a \equiv 0(r)}} \mu^2(n) \mu^2(n+a) 1_{(n,a)=1} = E(x; a) \prod_{p|qr} \frac{1}{p} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p^2}\right)^{-1} + O_a \left(x^{\frac{2}{3}+\epsilon} \tau([q, r])\right),$$

where here

$$E(x; a) := E(x) \prod_{p^2 || a} (1 + 1/p^2) \left(1 - \frac{1}{p} (1 - 1/p) (1 - 2/p^2)^{-1}\right).$$

*Proof.* By the Chinese remainder theorem, the pair of simultaneous congruences  $n \equiv 0(q)$ ,  $n \equiv -d(r)$  corresponds to the single congruence  $n \equiv c([q, r])$ , where  $c := \frac{q}{(q, r)} r \bar{r} - d \frac{q}{(q, r)} \bar{q} \in \mathbb{Z}/[q, r]\mathbb{Z}$ ; here,  $r \bar{r} \equiv 1(q/(q, r))$  and  $\frac{q}{(q, r)} \bar{q} \equiv 1(r)$ . Note that  $[q, r]$  is squarefree, and moreover  $p | ([q, r], c)$  if, and only if,  $p | q$ , and similarly,  $p | ([q, r], c + d)$  if, and only if,  $p | r$ . Applying the previous lemma with these observations implies (21).

For (22), we note that  $[q, r, e] = qre$  whenever  $(n, a) = 1$  and  $n \equiv 0(q)$ . Consequently,

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv 0(q), n+a \equiv 0(r)}} \mu^2(n) \mu^2(n+a) 1_{(n,a)=1} &= \sum_{e|a} \mu(e) \sum_{\substack{n \leq x \\ n \equiv 0(qe), n+a \equiv 0(r)}} \mu^2(n) \mu^2(n+a) \\ &= E(x) \prod_{p_1^2 || a} \left(1 + \frac{1}{p_1^2}\right) \sum_{e|a} \mu(e) \prod_{p_2 | qer} \frac{1}{p_2} \left(1 - \frac{1}{p_2}\right) \left(1 - \frac{2}{p_2^2}\right)^{-1} + O \left(x^{\frac{2}{3}+\epsilon} \tau(qr) \tau(a)\right) \\ &= E(x) \prod_{p_1^2 || a} \left(1 + \frac{1}{p_1^2}\right) \prod_{p_2 | qr} \frac{1}{p_2} \left(\frac{1 - 1/p_2}{1 - 2/p_2^2}\right) \sum_{e|a} \mu(e) \prod_{p_3 | e} \frac{1}{p_3} \left(\frac{1 - 1/p_3}{1 - 2/p_3^2}\right) + O \left(x^{\frac{2}{3}+\epsilon} \tau(qr) \tau(a)\right) \\ &= E(x; a) \prod_{p|qr} \frac{1}{p} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p^2}\right)^{-1} + O_a \left(x^{\frac{2}{3}+\epsilon} \tau(qr)\right), \end{aligned}$$

as claimed.  $\square$

We may now prove that the set of squarefree integers is siftable and regular.

*Proof of Proposition 2.2.* Referring to Definition 2.1, i) and ii) hold by Lemma 4.2 where we clearly see that  $R(x; q, r) \ll x^{\frac{2}{3}+\epsilon} \tau(qr) f(qr)^{-1}$ , with

$$(23) \quad f(p) := \frac{1}{p} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p^2}\right)^{-1} \leq \frac{1}{p}.$$

Moreover, note that if  $\theta \in (0, 1/6 - 2\epsilon)$  then if  $qr \leq x^\theta$ ,

$$\sum_{qr \leq x^\theta} R(x; q, r) \ll x^{\frac{2}{3}+\epsilon} \sum_{qr \leq x^\theta} (qr)^{1+\epsilon} \ll x^{\frac{2}{3}+2\theta+2\epsilon} \ll x^{1-2\epsilon},$$

so iii) holds as well with  $\theta = 1/7$ , say. Note that (23) and Mertens' theorem implies that  $\sum_{p \leq x} f(p) = \log_2 x + O(1)$ , so  $S$  is also regular.  $\square$

Theorem 1.2 now follows.

*Proof of Theorem 1.2.* We first observe that  $(n, a) = (n + a, a)$ . As such, we can write  $\mu_y(n)\mu_y(n + a) = \mu_y(n/(n, a))\mu_y((n + a)/(n, a))$ , so that

$$(24) \quad \sum_{n \leq x} \mu_y(n)\mu_y(n + a) = \sum_{\alpha|a} \sum_{m \leq x/\alpha} \mu_y(m)\mu_y(m + a/\alpha)1_{(m, a/\alpha)=1}.$$

Now, if  $\alpha$  is fixed  $b := a/\alpha$  and  $x' := x/\alpha$  then

$$\begin{aligned} & \left| \sum_{n \leq x'} \mu_y(n)\mu_y(n + b)1_{(n, b)=1} \right| = \left| \sum_{n \leq x} \mu^2(n)\mu^2(n + b)1_{(n, b)=1} e^{\pi i \omega_y(n)} e^{\pi i \omega_y(n+b)} \right| \\ &= \left| \sum_{n \leq x'} \mu^2(n)\mu^2(n + b)1_{(n, b)=1} e^{\pi i \sqrt{\log_2 y} \tilde{\omega}_y(n)} e^{\pi i \sqrt{\log_2 y} \tilde{\omega}_y(n+b)} \right| \\ &= E(x'; b) |\phi_{x', y}(\pi \sqrt{\log_2 y}, \pi \sqrt{\log_2 y})|. \end{aligned}$$

In light of Proposition 2.2, we may apply Proposition 3.6, yielding

$$x^{-1} \sum_{n \leq x'} \mu_y(n)\mu_y(n + b)1_{(n, b)=1} \ll_a \frac{1}{\log_2 y} + e^{-\frac{1}{21}\beta \log \beta} + \log^{-\frac{1}{6}} x.$$

Inserting this into (24) for each of the finitely many divisors  $\alpha$  of  $a$  completes the proof of (4). Equation (5) follows immediately from (19) and the triangle inequality.  $\square$

## 5. A DISJUNCTION THEOREM FOR CHARACTERISTIC FUNCTIONS

In this section, we show that if the distance between the characteristic function of a bivariate Gaussian random vector and that of a given random vector is not small then the  $L^\infty$  distance between the corresponding distribution functions of these random vectors is "smaller than expected". In the final section of this paper we will leverage this fact to gain better control on the difference between  $\phi_{x, y}(u, v)$  and  $\chi(u, v)$ , in the situation that  $S$  is the set of squarefree integers. To begin with, we will need the following arithmetic estimate that is relevant to this situation.

**Lemma 5.1.** *Let  $C > 0$ . Let  $k \in \mathbb{N}$  such that  $|k - \log_2 x| \leq C\sqrt{\log_2 x}$ . Then*

$$\tilde{\pi}_k(x) := |\{n \leq x : \mu^2(n(n+1)) = 1, \omega(n) = k\}| \gg_C \frac{x}{\sqrt{\log_2 x}}.$$

*Proof.* Let  $Y \geq 2$ . Given a modulus  $q$  and a residue class  $a$  modulo  $q$ , write  $\pi_k(x; a, q)$  to denote the number of integers  $n \leq x$  such that  $\omega(n) = k$  and  $n \equiv a(q)$ . Then we have

$$\begin{aligned} \tilde{\pi}_k(x) &= \sum_{\substack{n \leq x \\ \omega(n)=k}} \mu^2(n(n+1)) = \sum_{\substack{e_1 e_2 \leq Y \\ (e_1, e_2)=1}} \mu(e_1)\mu(e_2)\pi_k(x; (e_1 e_2)^2, c(e_1, e_2)) \\ &\quad + O\left(\sum_{\substack{e_1, e_2 \leq x^{\frac{1}{2}} \\ e_1 e_2 > Y}} \pi_k(x; (e_1 e_2)^2, c(e_1, e_2))\right) \\ &=: T_1 + T_2, \end{aligned}$$

where  $c(e_1, e_2)$  is the residue class modulo  $(e_1^2 e_2^2)$  corresponding to the pair of congruences  $n \equiv 0(e_1^2)$  and  $n \equiv -1(e_2^2)$ . Put  $\rho := \frac{k-1}{\log_2 x}$ . Fix  $e_1$  and  $e_2$  momentarily, and let  $q := (e_1 e_2)^2$ . By Theorem 2 of [20], the asymptotic formula

$$\pi_k(x; q, c(e_1, e_2)) = \frac{1}{\phi(q)} \frac{x(\log_2 x)^{k-1}}{(k-1)! \log x} \left( \frac{\alpha(\rho; q)}{\Gamma(1+\rho)} \left( \frac{\phi(q)}{q} \right)^\rho + O_\tau \left( \frac{\rho}{\log_2 x} (\log_4 x)^2 \right) \right)$$

holds uniformly in  $q \leq \log^\tau x$ , for each fixed  $\tau > 0$  and  $1 \leq k \leq 2 \log_2 x$ . Here, we put

$$\alpha(\rho; q) := \prod_{p \nmid q} \left( 1 + \frac{\rho(1-\rho)}{p(p-1)} + \rho(1-\rho)h_p(\rho) \right),$$

where the numbers  $h_p(\rho)$  satisfy  $\sum_p h_p(\rho) \ll 1$ . Choose  $Y = \log^6 x$  and put  $\rho = 1 + O\left(\frac{1}{\sqrt{\log_2 x}}\right)$ . Since

$$\alpha(\rho; q) = \exp \left( O \left( \rho(1-\rho) \sum_{p \nmid q} \frac{1}{p(p-1)} \right) \right) = 1 + O(|\rho - 1|) = 1 + O\left(\frac{1}{\sqrt{\log_2 x}}\right),$$

and also

$$(\phi(q)/q)^\rho = \frac{\phi(q)}{q} \left( 1 + O\left(\frac{\log_2 q}{\sqrt{\log_2 x}}\right) \right) = \frac{\phi(q)}{q} \left( 1 + O\left(\frac{\log_3 x}{\sqrt{\log_2 x}}\right) \right),$$

we have

$$\begin{aligned} T_1 &= \frac{x(\log_2 x)^{k-1}}{(k-1)! \log x} \left( \sum_{(e_1, e_2)=1} \frac{\mu(e_1)\mu(e_2)}{e_1^2 e_2^2} + O\left(\frac{\log_3 x}{\sqrt{\log_2 x}}\right) \right) \\ &= \left( 1 + O\left(\frac{\log_3 x}{\sqrt{\log_2 x}}\right) \right) \frac{E(x)(\log_2 x)^{k-1}}{(k-1)! \log x}. \end{aligned}$$

By Stirling's formula, we have

$$\begin{aligned} \frac{(\log_2 x)^{k-1}}{(k-1)! \log x} &= \left( 1 + O\left(\frac{1}{\log_2 x}\right) \right) (2\pi)^{-\frac{1}{2}} \left( \frac{e \log_2 x}{k-1} \right)^{k-1} e^{-\log_2 x} \frac{1}{\sqrt{\log_2 x}} \\ &= \left( 1 + O\left(\frac{1}{\log_2 x}\right) \right) (2\pi)^{-\frac{1}{2}} e^{k-1-\log_2 x} \cdot \left( 1 + \frac{k-1-\log_2 x}{\log_2 x} \right)^{-(k-1)} \frac{1}{\sqrt{\log_2 x}} \\ &\gg_C \frac{1}{\sqrt{\log_2 x}}, \end{aligned}$$

so that  $T_1 \gg_C x(\log_2 x)^{-\frac{1}{2}}$ . It now suffices to show that  $T_2 = o\left(\frac{x}{\sqrt{\log_2 x}}\right)$  to complete the proof.

For the sum in  $T_2$ , we simply bound the terms  $\pi_k(x; (e_1 e_2)^2, c(e_1, e_2))$  by the number of integers  $n \leq x$  satisfying  $\mu^2(n(n+1)) = 1$  with  $n \equiv c(e_1^2 e_2^2)$  and apply the bound in the proof of Lemma 4.1 (with  $q = 1$  there), getting

$$T_2 \ll xY^{-\frac{1}{2}} \log^2 x \ll x \log^{2-\tau/2} x \ll \frac{x}{\log x}.$$

The claim follows.  $\square$

We surmise from the Lemma 5.1 the minimum order of growth of the  $L^\infty$  distance between  $F_{x,y}$  and  $\Phi_{(2)}$ , where  $F_{x,y}$  is the distribution function whose characteristic function is  $\phi_{x,y}$ .

**Proposition 5.2.** *Let  $C > 0$  be fixed. Let  $y := x^{\frac{1}{\beta}}$  with  $\beta \leq e^{B(x)}$ . Let  $\mathfrak{L}_y := \sqrt{\log_2 y}$ ,*

$$F_{x,y}(t, t') := E(x)^{-1} |\{n \leq x : \mu^2(n(n+1)) = 1, \tilde{\omega}_y(n) \leq t, \tilde{\omega}_y(n+1) \leq t'\}|.$$

*Then the estimate*

$$(25) \quad \|F_{x,y} - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} = o(\mathfrak{L}^{-1})$$

*cannot hold.*

*Proof.* Assume that (25) holds. Let  $t = \log_2 y$ . By the Turán-Kubilius inequality (see Section III.2 in [23]), for any  $t' \in \mathbb{R}$  we have

$$|F_{x,y}(t', t) - F_{x,y}(t', \infty)| \ll x^{-1} |\{n \leq x : |\omega_y(n+1) - \log_2 y| > \log_2 y\}| \ll \frac{1}{\log_2 y}.$$

Put now  $t' = 0$ . Since  $\Phi(t) \ll 1/\log x$ ,

$$\begin{aligned} & F_{x,y}(t' + (2\mathfrak{L}_y)^{-1}, \infty) - F_{x,y}(t' - (2\mathfrak{L}_y)^{-1}, \infty) \\ &= F_{x,y}(t' + (2\mathfrak{L}_y)^{-1}, t) - F_{x,y}(t' - (2\mathfrak{L}_y)^{-1}, t) + O(\mathfrak{L}_y^{-2}) \\ &= \Phi(t) (\Phi(t' + (2\mathfrak{L}_y)^{-1}) - \Phi(t' - (2\mathfrak{L}_y)^{-1})) + O(\mathfrak{L}^{-2} + \|F_{x,y} - G\|_{L^\infty(\mathbb{R}^2)}) = o(\mathfrak{L}^{-1}). \end{aligned}$$

On the other hand, by Lemma 5.1,

$$\begin{aligned} & F_{x,y}(t' + (2\mathfrak{L}_y)^{-1}, \infty) - F_{x,y}(t' - (2\mathfrak{L}_y)^{-1}, \infty) \\ &= E(x)^{-1} |\{n \leq x : \mu^2(n(n+1)) = 1, |\omega_y(n) - \log_2 y| < 1\}| \\ &\geq E(x)^{-1} |\{n \leq x : \mu^2(n(n+1)) = 1, |\omega(n) - \log_2 x| < 1\}| \gg \mathfrak{L}^{-1}. \end{aligned}$$

This is an obvious contradiction.  $\square$

The last two lemmata imply the following disjunction result, which allows one to extract additional savings on the distance between  $\phi_{x,y}$  and the characteristic function of the bivariate Gaussian distribution.

**Proposition 5.3.** *Let  $Z, T \geq 1$ ,  $y = x^{\frac{1}{\beta}}$  with  $\beta \leq e^{B(x)}$  and  $u, v \in \mathbb{R}$ . Let  $F_{x,y}$ ,  $\Phi_{(2)}$ ,  $\phi_{x,y}$ ,  $\chi$  and  $\lambda(y)$  be as above. Then exactly one of the following holds:*

*i) we have*

$$|\phi_{x,y}(u, v) - \chi(u, v)| \ll Z \left( \frac{1}{T} + \int_{[T^{-3}, T]^2} \frac{|\phi_{x,y}(t, t') - \chi(t, t')|}{|tt'|} dt dt' \right);$$

*ii) there is a bivariate Poisson random vector  $X = (X_1, X_2) \sim \text{Poi}(\lambda(y))^2$  defined on  $[1, x]$  with  $X_1, X_2$  independent, such that if  $\tilde{X}_j := (X_j - \lambda(y))/\sqrt{\lambda(y)}$  and  $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$  then*

$$\|F_{x,y} - \Phi_{(2)}\|_{L^\infty(\mathbb{R}^2)} \ll Z^{-1} \left( d_{TV}(\mathcal{L}(\tilde{\omega}_y, \tilde{\omega}_y \circ T), \mathcal{L}(\tilde{X})) + e^{-\frac{1}{2}w^2} w^4 (\log_2 y)^{-1} \right).$$

*Proof.* Suppose first that  $|\phi_{x,y}(u, v) - \chi(u, v)| \leq Z \|F_{x,y} - \Phi_{(2)}\|_\infty$ . Then i) follows upon applying Lemma 3.1 together with Corollary 3.3.

Suppose now that  $\|F_{x,y} - G\|_\infty < Z^{-1} |\phi_{x,y}(u, v) - \chi(u, v)|$ . Write  $\lambda$  in place of  $\lambda(z)$  for convenience, and let  $X = (X_1, X_2)$  be as in the statement. Thus,

$$\begin{aligned} \|F_{x,y} - \Phi_{(2)}\|_\infty &< Z^{-1} (|\phi_{x,y}(u, v) - \pi_y(u)\pi_y(v)| + |\pi_y(u)| |\pi_y(v) - \chi(v)| + |\chi(v)| |\pi_y(u) - \chi(u)|) \\ &\ll Z^{-1} \left( |\phi_{x,y}(u, v) - \pi_y(u)\pi_y(v)| + \max_{w \in \{u, v\}} |\pi_y(w) - \chi(w)| \right). \end{aligned}$$

By Lemma 3.4,  $|\phi_{x,y}(u, v) - \pi_y(u)\pi_y(v)| \ll d_{TV}(\mathcal{L}(\tilde{\omega}_y, \tilde{\omega}_y \circ T), \mathcal{L}(\tilde{X}))$ . Furthermore, the proof of Proposition 3.6 shows that

$$|\pi_y(w) - \chi(w)| \leq e^{-\frac{1}{2}w^2} w^4 (\log_2 y)^{-1}.$$

This implies the claim.  $\square$

## 6. ON A PROBLEM OF ERDŐS AND MIRSKY

We begin this final section by explaining a version of the Bateman-Spiro heuristic underlying the statement of Conjecture 1.10. Recall that

$$S(x) := \{n \leq x : \tau(n) = \tau(n+1)\}.$$

In light of the result in [6] it suffices to show that  $S(x) \gg x(\log_2 x)^{-\frac{1}{2}}$ . Let

$$S^*(x) := |\{n \leq x : \mu^2(n(n+1)) = 1, \tau(n) = \tau(n+1)\}|.$$

Clearly,  $S(x) \geq S^*(x)$ , and we seek to bound  $S^*(x)$  from below. By Lemmata 6.3 and 6.4 below (neither of which are deep) we have

$$(26) \quad S^*(x) \gg \frac{E(x)}{\log_2 x} \left| \int_0^{2\pi\sqrt{\log_2 x}} \int_0^{2\pi\sqrt{\log_2 x}} \phi_x(u - u', u' - u) du du' \right|.$$

Suppose that we had some control over the error term  $|\phi_x(v, -v) - \chi(v, -v)|$  in such a way that this term is negligible for each  $v \in [-2\pi\sqrt{\log_2 x}, 2\pi\sqrt{\log_2 x}]$ . Then we could replace  $\phi_x(u - u', u' - u)$  in (26) by  $e^{-(u-u')^2}$ . It is then easy to prove (see Lemma 6.5) that the corresponding integral is  $\gg \sqrt{\log_2 x}$ . When used in (26), this gives the conjectured lower bound  $\frac{x}{\sqrt{\log_2 x}}$  for  $S^*(x)$  on heuristic grounds.

To furnish the lower bound rigorously, we would need to show that we can suitably bound  $|\phi_x(u - u', u' - u) - \chi(u - u', u' - u)|$ , at least in an average sense, in the box  $[0, 2\pi\sqrt{\log_2 x}]^2$ . As we cannot do this, we settle for the following analogue of Conjecture 1.10.

Let  $2 \leq y \leq x$ , such that if  $\beta := (\log x)/(\log y)$  then  $A(x) \leq \beta \leq e^{B(x)}$ . We note, in particular, that  $\log_2 y = (1 + o(1)) \log_2 x$ . Write  $\mathfrak{L}_y := \sqrt{\log_2 y}$  and  $\mathfrak{L} := \mathfrak{L}_x$  as before. For  $|j| \leq (1 - \epsilon) \sqrt{\log_2 x \log_3 x}$  let

$$\begin{aligned} S_j(x; y) &:= |\{n \leq x : \tau_y(n) = 2^j \tau_y(n+1)\}| \\ S_j^*(x; y) &:= |\{n \leq x : \mu^2(n(n+1)) = 1, \tau_y(n) = 2^j \tau_y(n+1)\}|, \end{aligned}$$

where  $\tau_y$  is the  $y$ -smooth divisor function, i.e.,  $\tau_y(n) := \prod_{p^k || n} (k+1)$ . We will estimate  $S_j^*$  from below to give a lower bound for  $S_j$ .

**Theorem 6.1.** *Let  $2 \leq y \leq x$  such that if  $y = x^{\frac{1}{\beta}}$  then  $A(x) \leq \beta \leq e^{B(x)}$ . Then for each  $|j| \leq (1 - \epsilon)\sqrt{\log_2 x \log_3 x}$ , we have*

$$|\{n \leq x : \tau_y(n) = 2^j \tau_y(n+1)\}| \gg e^{-\frac{j^2}{4\mathcal{L}_y^2}} \frac{x}{\sqrt{\log_2 x}}.$$

**Remark 6.2.** Note that the set of  $n \leq x$  such that  $\tau(n) = \tau(n+1)$  distributes among sets in which  $\tau_y(n) = 2^j \tau_y(n+1)$ , for some  $|j| \leq \frac{\log x}{\log y}$ . However, most integers have about  $\log\left(\frac{\log x}{\log y}\right)$  such factors, and we thus expect that  $n$  and  $n+a$  should have nearly the same number of prime factors of size larger than  $y$ . Thus, the restriction on  $j$  here is not expected to be problematic when  $y$  is close to the upper limit of its range. If we had a better understanding of which  $j$  occur most often such that  $\tau(n) = \tau(n+1)$  and  $\tau_y(n) = 2^j \tau_y(n+1)$  then we would have a chance at actually proving Conjecture 1.10.

The relevance of the results of the previous sections to this problem is brought to light via the following lemma.

**Lemma 6.3.** *Let  $x \geq 3$  and  $\alpha > 0$ . Then*

$$S_j^*(x; y) = \frac{E(x)}{(\pi \mathcal{L}_y \log 2)^2} \int_{[0, 2\pi \mathcal{L}_y]^2} g_{y,\alpha}(u) g_{y,\alpha}(u') \phi_{x,y}(u - u', u' - u) e^{i(j/\mathcal{L}_y)(u' - u)} d\mathbf{u},$$

where we set

$$\begin{aligned} g_{y,\alpha}(u) &:= \lim_{M \rightarrow \infty} \sum_{|k| \leq M} \frac{1}{\alpha + i(2\pi k + u)/\log 2} \\ &= \frac{1}{\alpha + iu/\mathcal{L}_y} + \sum_{k \geq 1} \frac{\alpha + iu/(\mathcal{L}_y \log 2)}{(\alpha + iu/(\mathcal{L}_y \log 2))^2 + 4\pi^2 k^2 / \log^2 2}. \end{aligned}$$

*Proof.* Let  $b > 0$ . Applying the Mellin identity

$$(27) \quad \frac{1}{2\pi i} \int_{(\alpha)} b^s \frac{ds}{s} = \lim_{T \rightarrow \infty} \left( \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} b^s \frac{ds}{s} \right) = \begin{cases} 1 & \text{if } b > 1 \\ \frac{1}{2} & \text{if } b = 1 \\ 0 & \text{if } b < 1 \end{cases}$$

for  $b = \tau(n)/(2^j \tau(n+1))$  and its reciprocal yields

$$(\pi i)^{-2} \lim_{T, T' \rightarrow \infty} \left( \int_{\alpha - iT}^{\alpha + iT} \int_{\alpha - iT'}^{\alpha + iT'} \frac{ds ds'}{ss'} \left( \frac{\tau(n)}{2^j \tau(n+1)} \right)^s \left( \frac{2^j \tau(n+1)}{\tau(n)} \right)^{s'} \right) = \begin{cases} 1 & \text{if } \frac{\tau(n)}{\tau(n+1)} = 2^j \\ 0 & \text{otherwise.} \end{cases}$$



Summing this identity over squarefree integers  $n \leq x$ , we have

$$\begin{aligned}
S_j^*(x; y) &= \frac{1}{\pi^2} \lim_{T, T' \rightarrow \infty} \left( \int_{-T}^T \int_{-T'}^{T'} \frac{dt dt'}{(\alpha + it)(\alpha + it')} \sum_{\substack{n \leq x \\ \mu^2(n(n+1))=1}} \tau_y(n)^{i(t-t')} \tau_y(n+1)^{i(t'-t)} 2^{ij(t'-t)} \right) \\
&= \frac{1}{\pi^2} \lim_{T, T' \rightarrow \infty} \left( \int_{-T}^T \int_{-T'}^{T'} \frac{dt dt'}{(\alpha + it)(\alpha + it')} \sum_{\substack{n \leq x \\ \mu^2(n(n+1))=1}} e^{i\omega_y(n)(t-t') \log 2} e^{i\omega_y(n+1)(t'-t) \log 2} e^{ij(t'-t) \log 2} \right) \\
&= \frac{1}{(\pi \log 2)^2} \lim_{T, T' \rightarrow \infty} \left( \int_{-T}^T \int_{-T'}^{T'} \frac{dt dt'}{(\alpha + \frac{it}{\log 2})(\alpha + \frac{it'}{\log 2})} \sum_{\substack{n \leq x \\ \mu^2(n(n+1))=1}} e^{i(t-t')\omega_y(n)} e^{i(t'-t)\omega_y(n+1)} e^{ij(t'-t)} \right).
\end{aligned}$$

We now observe that the sum over  $n$  is invariant as a function of  $t$  and  $t'$  under the translations  $(t, t') \mapsto (t - 2\pi k, t - 2\pi l)$ , for  $k, l \in \mathbb{Z}$ . Since the double limit defining  $S_j^*(x, y)$  exists, we may rewrite it as

$$\begin{aligned}
S_j^*(x; y) &= \frac{1}{(\pi \log 2)^2} \lim_{\substack{M, N \rightarrow \infty \\ M, N \in \mathbb{Z}}} \left( \int_{-2\pi M}^{2\pi M} \int_{-2\pi N}^{2\pi N} \frac{dt dt'}{(\alpha + it/\log 2)(\alpha + it'/\log 2)} \right. \\
&\quad \cdot \sum_{\substack{n \leq x \\ \mu^2(n(n+1))=1}} e^{i(t-t')\omega_y(n)} e^{i(t'-t)\omega_y(n+1)} e^{ij(t'-t)} \Big) \\
&= \frac{1}{(\pi \log 2)^2} \int_{[0, 2\pi]^2} \left( \sum_{n \leq x} \mu^2(n(n+1)) e^{i(t-t')\omega_y(n)} e^{i(t'-t)\omega_y(n+1)} e^{ij(t'-t)} dt dt' \right) \\
&\quad \cdot \lim_{\substack{M, N \rightarrow \infty \\ M, N \in \mathbb{Z}}} \left( \sum_{|k| \leq M} \frac{1}{\alpha + i(t + 2\pi k)/\log 2} \right) \left( \sum_{|l| \leq N} \frac{1}{\alpha + i(t' + 2\pi l)/\log 2} \right).
\end{aligned}$$

We may simplify the sum over  $k$  via

$$\begin{aligned}
\sum_{|k| \leq M} \frac{1}{\alpha + i(t + 2\pi k)/\log 2} &= \frac{1}{\alpha + it/\log 2} + \sum_{1 \leq k \leq M} \left( \frac{1}{\alpha + i(t + 2\pi k)/\log 2} + \frac{1}{\alpha + i(t - 2\pi k)/\log 2} \right) \\
&= \frac{1}{\alpha + it/\log 2} + 2 \sum_{1 \leq k \leq M} \frac{\alpha + it/\log 2}{(\alpha + it/\log 2)^2 + 4\pi^2 k^2 / \log^2 2}.
\end{aligned}$$

A similar expression exists for the sum over  $l$  (as a function of  $t'$  instead of  $t$ ). Since each of these sums converges absolutely and uniformly in  $[0, 2\pi]$ , we may take  $M, N \rightarrow \infty$  to give

$$S_j^*(x; y) = \frac{1}{(\pi \log 2)^2} \int_{[0, 2\pi]^2} \left( \sum_{\substack{n \leq x \\ \mu^2(n(n+1))=1}} e^{i(t-t')\mathfrak{L}_y \tilde{\omega}_y(n)} e^{i(t'-t)\mathfrak{L}_y \tilde{\omega}_y(n+1)} \right) e^{ij(t'-t)} g_{y, \alpha}(t \mathfrak{L}_y) g_{y, \alpha}(t' \mathfrak{L}_y) dt.$$

Making the change of variables  $u := t \mathfrak{L}_y$  and  $u' := t' \mathfrak{L}_y$  completes the proof of the lemma.  $\square$

We will choose  $\alpha$  suitably so that  $g_{y,\alpha}$  is well-behaved. The following lemma gives us a hint in this direction.

**Lemma 6.4.** *Uniformly in  $[0, 2\pi\mathfrak{L}_y]$ , we have  $g_{y,\alpha}(u) = \frac{1}{4} + O\left(\frac{1}{\alpha}\right)$ , as  $\alpha \rightarrow \infty$ .*

*Proof.* Note that  $(\alpha + iu/\mathfrak{L}_y)^{-1} = \frac{1}{\alpha} \left(1 + O\left(\frac{1}{\alpha}\right)\right)$  uniformly, since  $|u/\mathfrak{L}_y|$  is uniformly bounded. For the series, we have

$$\begin{aligned} & \sum_{k \geq 1} \frac{\alpha + iu/(\mathfrak{L}_y \log 2)}{\alpha^2 - u^2/(\mathfrak{L}_y \log 2)^2 + 4\pi^2 k^2 + 2i\alpha u/(\mathfrak{L}_y \log 2)} \\ &= \alpha \left(1 + O\left(\frac{1}{\alpha}\right)\right) \sum_{k \geq 1} \frac{1}{4\pi^2 k^2 + (\alpha^2 - u^2/(\mathfrak{L}_y \log 2)^2)} \\ &= \alpha \left(1 + O\left(\frac{1}{\alpha}\right)\right) \sum_{k \geq 1} \frac{1}{4\pi^2 k^2 + \alpha^2}. \end{aligned}$$

Approximating the latter sum by an integral, we have

$$\begin{aligned} \sum_{k \geq 1} \frac{1}{4\pi^2 k^2 + \alpha^2} &= \frac{1}{2\pi\alpha} \left( \frac{2\pi}{\alpha} \sum_{k \geq 1} \frac{1}{1 + \left(\frac{2\pi k}{\alpha}\right)^2} \right) \\ &= \frac{1}{2\pi\alpha} \int_0^\infty \frac{dt}{1+t^2} + \frac{1}{2\pi\alpha} \left( \frac{2\pi}{\alpha} \sum_{k \geq 1} \int_k^{k+1} \left( \frac{1}{1 + \left(\frac{2\pi k}{\alpha}\right)^2} - \frac{1}{1 + \left(\frac{2\pi t}{\alpha}\right)^2} \right) dt \right) \\ &= \frac{1}{4\alpha} + O\left( \sum_{k \geq 1} \frac{k}{(\alpha^2 + 4\pi^2 k^2)^2} \right). \end{aligned}$$

We estimate the error term by splitting the sum into the ranges  $k \leq \alpha/2\pi$  and its complement. In the first range,

$$\sum_{k \leq \alpha/2\pi} \frac{k}{(\alpha^2 + 4\pi^2 k^2)^2} \ll \frac{1}{\alpha^4} \sum_{k \leq \alpha/2\pi} k \ll \frac{1}{\alpha^2}.$$

In the second range,

$$\sum_{k > \alpha/2\pi} \frac{k}{(\alpha^2 + 4\pi^2 k^2)^2} \ll \sum_{k > \alpha/2\pi} \frac{1}{k^3} \ll \frac{1}{\alpha^2}.$$

It follows from this that  $g_{y,\alpha}(u) = \frac{1}{4} + O\left(\frac{1}{\alpha}\right)$  uniformly in the interval, and the proof is complete.  $\square$

As a consequence, we have the following.

**Lemma 6.5.** *There is an  $\alpha_0 > 0$  such that for  $\alpha \geq \alpha_0$ ,*

(28)

$$\int_{[0, 2\pi\mathfrak{L}_y]^2} g_{y,\alpha}(u) g_{y,\alpha}(u') e^{i(j/\mathfrak{L}_y)(u'-u)} e^{-(u-u')^2} d\mathbf{u} = \left( \frac{\pi^{\frac{3}{2}}}{8} + O\left(\frac{1}{\alpha}\right) \right) e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \mathfrak{L}_y + O\left( e^{\frac{j^2}{4\mathfrak{L}_y^2}} \right).$$

*Proof.* By Lemma 6.4 we have  $g_{y,\alpha}(u)g_{y,\alpha}(u') = \frac{1}{16} + O\left(\frac{1}{\alpha}\right)$  for sufficiently large  $\alpha$ . Thus, it suffices to show that

$$\int_{[0, 2\pi\mathfrak{L}_y]^2} e^{-(u-u')^2 - i(j/\mathfrak{L}_y)(u-u')} d\mathbf{u} = 2\pi^{\frac{3}{2}} e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \mathfrak{L}_y + O\left(e^{\frac{j^2}{4\mathfrak{L}_y^2}}\right).$$

Making a change of variables, the integral on the right may be written as

$$\int_{[0, 2\pi\mathfrak{L}_y]^2} e^{-(u-u')^2 - i(j/\mathfrak{L}_y)(u-u')} d\mathbf{u} = e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \int_0^{2\pi\mathfrak{L}_y} du' \left( \int_{-u'}^{u'} + \int_{u'}^{2\pi\mathfrak{L}_y - u'} \right) e^{-\left(v - i\frac{j}{2\mathfrak{L}_y}\right)^2} dv.$$

We note that by standard contour integration,

$$\int_{-u'}^{u'} e^{-\left(v - i\frac{j}{2\mathfrak{L}_y}\right)^2} dv = \sqrt{\pi} - \int_{|v| > u'} e^{-\left(v - i\frac{j}{2\mathfrak{L}_y}\right)^2} dv = \sqrt{\pi} \left( 1 + O\left(e^{-\frac{1}{2}(u')^2} e^{\frac{j^2}{4\mathfrak{L}_y^2}}\right) \right),$$

so that

$$\int_0^{2\pi\mathfrak{L}_y} du' \int_{-u'}^{u'} e^{-v^2} dv = 2\pi^{\frac{3}{2}} \mathfrak{L}_y + O\left(e^{j^2/4\mathfrak{L}_y^2} \int_0^\infty e^{-\frac{1}{2}(u')^2} du'\right) = 2\pi^{\frac{3}{2}} \mathfrak{L}_y + O\left(e^{\frac{j^2}{4\mathfrak{L}_y^2}}\right).$$

In a similar vein, we have

$$\int_{u'}^{2\pi\mathfrak{L}_y - u'} e^{-v^2} dv \leq \int_{|v| > u'} e^{-v^2} dv \ll e^{-\frac{1}{2}(u')^2},$$

so that

$$\int_0^{2\pi\mathfrak{L}_y} du' \int_{u'}^{2\pi\mathfrak{L}_y - u'} e^{-v^2} dv \ll \int_0^\infty e^{-\frac{1}{2}(u')^2} du' \ll 1.$$

This implies the claim.  $\square$

*Proof of Theorem 6.1.* As before, we split

$$\phi_x(u - u', u' - u) = \chi(u - u', u' - u) + (\phi_x(u - u', u' - u) - \chi(u - u', u' - u)).$$

Thus,

$$S_j^*(x; y)/E(x) = \mathcal{M} + \mathcal{E},$$

where we put

$$\mathcal{M} := \frac{1}{(\pi\mathfrak{L}_y \log 2)^2} \int_{[0, 2\pi\mathfrak{L}_y]^2} g_x(u)g_x(u') e^{-(u-u')^2 - i(j/\mathfrak{L}_y)(u-u')} d\mathbf{u},$$

$$\mathcal{E} \ll \frac{1}{\mathfrak{L}_y^2} \int_0^{2\pi\mathfrak{L}_y} \int_0^{2\pi\mathfrak{L}_y} \left| E(x)^{-1} \sum_{\substack{n \leq x \\ \mu^2(n(n+1))=1}} e^{i(u-u')\tilde{\omega}_y(n)} e^{i(u'-u)\tilde{\omega}_y(n+1)} - e^{-(u-u')^2} \right| d\mathbf{u}.$$

By Lemma 6.5,  $\mathcal{M} \asymp e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \mathfrak{L}_y^{-1}$  when  $|j| \leq (1 - \epsilon) \sqrt{\log_2 y \log_3 x}$ . Our objective now is to show that  $\mathcal{E} = o\left(e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \mathfrak{L}_y^{-1}\right)$ . Making a change of variables as above and using

symmetry,

$$\begin{aligned} \mathcal{E} &\ll \mathfrak{L}^{-2} \int_0^{\pi \mathfrak{L}_y} du' \int_{-u'}^{2\pi \mathfrak{L}_y - u'} \left| \phi_{x,y}(v, -v) - e^{-v^2} \right| dv \\ &\leq \mathfrak{L}^{-1} \max_{u' \in [0, \pi \mathfrak{L}_y]} \int_{-u'}^{2\pi \mathfrak{L}_y - u'} \left| \phi_{x,y}(v, -v) - e^{-v^2} \right| dv. \end{aligned}$$

To estimate  $\mathcal{E}$  we use Propositions 5.2 and 5.3. Indeed, combining the proof of the latter proposition with Lemma 2.8, we see if we choose  $Z$  such that

$$Z^{-1} \left( \frac{1}{\mathfrak{L}_y^2} + \left| \exp \left( -\mathfrak{L}_y^2 \left( 1 - \cos \left( \frac{v}{\mathfrak{L}_y} \right) \right) \right) - e^{-v^2} \right| \right) = o(\mathfrak{L}_y^{-1})$$

then by Proposition 5.2, ii) in Proposition 5.3 cannot hold, and thus for any  $T \geq 1$  we have

$$|\phi_x(v, -v) - \chi(v, -v)| \ll Z \left( \int_{[T^{-3}, T]^2} \frac{|\phi_{x,y}(t, t') - \chi(t, t')|}{|tt'|} dt dt' + T^{-1} \right).$$

(Here we have used the fact that  $e^{-\frac{1}{21}\beta \log \beta} \ll \mathfrak{L}^{-2}$ .) Inserting the estimates for the term in absolute value derived in the proof of Proposition 3.6 and noting that  $\mathfrak{L}_y \asymp \mathfrak{L}_x$ , we see that it suffices to choose  $Z$  such that

$$Z \asymp \begin{cases} (\log_4 x) \mathfrak{L}^{-1} & \text{if } |v| \leq \log_2^{\frac{1}{4}} x \\ (\log_4 x) \left( \mathfrak{L}^{-2} + \exp \left( -\mathfrak{L}_y^{\frac{3}{2}} \right) \right) & \text{otherwise.} \end{cases}$$

As such, according to the argument in the proof of Theorem 2.3, we get

$$|\phi_{x,y}(v, -v) - \chi(v, -v)| \ll \begin{cases} (\log_4 x)(\log_3 x)(\log_2 x)^{-\frac{1}{4}} \mathfrak{L}^{-1} & \text{if } |v| \leq \log_2^{\frac{1}{4}} x \\ (\log_4 x)(\log_3 x)(\log_2 x)^{-\frac{1}{4}} \left( \mathfrak{L}^{-2} + \exp \left( -\mathfrak{L}_y^{\frac{3}{2}} \right) \mathfrak{L} \right) & \text{otherwise.} \end{cases}$$

Inserting these estimates into our expression for  $\mathcal{E}$  gives

$$\begin{aligned} \mathcal{E} &\ll \mathfrak{L}^{-1} \left( \int_{|v| \leq \mathfrak{L}^{\frac{1}{2}}} + \int_{|v| > \mathfrak{L}^{\frac{1}{2}}} \right) |\phi_{x,y}(v, -v) - \chi(v, -v)| dv \\ &\ll (\log_4 x) \left( \mathfrak{L}^{-1} (\log_3 x)(\log_2 x)^{-\frac{1}{4}} + \mathfrak{L}^2 \exp \left( -\mathfrak{L}_y^{\frac{3}{2}} \right) \right) \\ &\ll \mathfrak{L}^{-1} (\log_4 x)(\log_3 x)(\log_2 x)^{-\frac{1}{4}}. \end{aligned}$$

Since  $e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \gg (\log_4 x)(\log_3 x)(\log_2 x)^{-\frac{1}{4}}$  when  $|j| \leq (1 - \epsilon) \sqrt{\log_2 y \log_3 x}$ , we indeed have  $\mathcal{E} = o \left( e^{-\frac{j^2}{4\mathfrak{L}_y^2}} \mathfrak{L}^{-1} \right)$  in this case. Since  $\log_2 x = (1 + o(1)) \log_2 y$ , this completes the proof.  $\square$

#### ACKNOWLEDGMENTS

The author would like to thank his Ph.D supervisor John Friedlander, as well as Asif Zaman and Oleksiy Klurman for their encouragement and interest in the results of this paper. He is indebted to Youness Lamzouri for his helpful comments regarding how to improve the paper's exposition. Finally, the author thanks Maksym Radziwiłł and Sary Drappeau for providing additional references.

## REFERENCES

- [1] J. Cassaigne, S. Ferenczi, C. Mauduit, J. Rivat, and A. Sarkőzy. On finite pseudorandom binary sequences. iii. the Liouville function. i. *Acta Arithmetica*, 87(4):367–390, 1999.
- [2] S. Chowla. *The Riemann Hypothesis and Hilbert’s Tenth Problem*. Gordon and Breach Science Publishers, New York-London-Paris, 1965.
- [3] H. Daboussi and A. Sarkőzy. On the correlation of the truncated Liouville function. *Acta Arith.*, 108:61–76, 2003.
- [4] P.D.T.A. Elliott. *Probabilistic Number Theory I*. Springer-Verlag, New York, U.S.A, 1979.
- [5] P. Erdős and L. Mirsky. The distribution of values of the divisor function  $d(n)$ . *Proc. London Math. Soc.*, 3:257–271, 1952.
- [6] P. Erdős, C. Pomerance, and A. Sarkőzy. On locally repeated values of certain arithmetic functions III. *Proc. Amer. Math. Soc.*, 101:1–7, 1987.
- [7] W. Feller. *An Introduction to Probability Theory and its Applications Vol. 2*. Wiley Publishing, New York, NY, 1957.
- [8] J. Friedlander and H. Iwaniec. *Opera de Cribro*. American Mathematical Society, Providence, RI, 2010.
- [9] H. Ganguli. *On the Correlation of Completely Multiplicative Functions*. PhD thesis, Simon Fraser University, Vancouver, Canada, 2013.
- [10] E. Goudout. Lois locales de la fonction  $\omega$  dans presque tous les petits intervalles. arXiv:1607.08666 [math.NT].
- [11] A. Granville and K. Soundararajan. *Sieving and the Erdős-Kac theorem*, pages 15–27. Springer Netherlands, Dordrecht, 2007.
- [12] D.R. Heath-Brown. The square sieve and consecutive square-free numbers. *Mathematische Annalen*, 266:251–260, 1984.
- [13] A. Hildebrand. The divisor function at consecutive integers. *Pac. J. Math.*, 129:307–319, 1987.
- [14] J. Kubilius. *Probabilistic Methods in the Theory of Numbers*. AMS, Translations of Mathematical Monographs, Providence, RI, 1964.
- [15] W.J. Leveque. On the size of certain number-theoretic functions. *Trans. Amer. Math. Soc.*, 66:440–463, 1949.
- [16] K. Matomäki and M. Radziwiłł. Multiplicative functions in short intervals. *Ann. of Math*, 183:1015–1056, 2016.
- [17] K. Matomäki, M. Radziwiłł, and T. Tao. An averaged form of Chowla’s conjecture. *Algebra and Number Theory*, 9:2167–2196, 2015.
- [18] A. Renyi and P. Turán. On a theorem of Erdos-Kac. *Acta Arithmetica*, 4:71–84, 1958.
- [19] B. Roos. Metric multivariate Poisson approximation of the generalized multinomial distribution. *Teor. Veroyatnost. i Primenen*, 43:404–413, 1998.
- [20] C. Spiro. Extensions of some formulae of A. Selberg. *Internat. J. Math. Math. Sci.*, 8:283–302, 1985.
- [21] T. Tao. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. arXiv:1509.05422 [math.NT].
- [22] T. Tao. The Katai-Bourgain-Sarnak-Ziegler orthogonality criterion. <https://terrytao.wordpress.com/2011/11/21/the-bourgain-sarnak-ziegler-orthogonality-criterion/>, 2011.
- [23] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. Cambridge University Press, Cambridge, UK, 1994.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA  
 E-mail address: [sacha.mangerel@mail.utoronto.ca](mailto:sacha.mangerel@mail.utoronto.ca)